Committee 1
Symmetry in Its Various Aspects:
Search for Order in the Universe

Draft – February 1, 2000
For Conference Distribution Only

The Two Leonardos, Part I. Nature's Numbers: Leonardo Fibonacci di Pisa

Bulent I. Atalay
Professor of Physics
Mary Washington College and
University of Virginia
Fredericksburg, Virginia

The Twenty-second International Conference on the Unity of the Sciences
Seoul, Korea February 9-13, 2000

ABSTRACT

Part 1. Leonardo Fibonacci di Pisa: Nature's Numbers

Three types of regular polygons have always been identified as the shapes capable of mathematical tiling in two dimensions. Similarly, three types of regular polygons were found by Ancient Greek philosophers to create the five regular polyhedra of solid geometry. To the Pythagoreans four of these polyhedra represented the shapes of the *atoms* of nature. Symmetric shapes and regularities are observed well beyond the capabilities of our senses – at the ultra-microscopic scale as well as the supra-macroscopic, the former with electron diffraction and scanning tunneling microscopes; the latter, with optical and radio telescopes.

Along with a presentation of some of the regularities seen in nature, there will be a discussion of number systems with various bases, and the eventual appearance of the decimal system – replete with the zero – and the transmission of this system to Europe in the early Thirteenth Century. In his monumental book *the Liber Abacis*, Fibonacci introduced the celebrated series that bears his name (1, 1, 2, 3, 5, 8, ...), and that ubiquitous ratio issuing from the series, (ϕ=*1.618 034* ...) virtually as an after thought. The series is found to have significance in genetics, phyllotaxis, in mathematical mosaics, and in crystallography. But the remarkable ratio was seen in Man's artistic creations as and intuitive scheme long before the publication of the Fibonacci Series.

SYMMETRY IN ART AND NATURE: THE TWO LEONARDOS

In nature we observe symmetric shapes at the macroscopic level both in animate and in inanimate objects. At the microscopic level beyond the capabilities of our natural senses, and at the supra macroscopic, some of the same shapes, symmetries, and regularities prevail. The cross-section of the micro tubules in the heliozoan, magnified one-hundred thousand times, displays the same spiral shapes as do the horns of the ram, and multiplied another hundred billion billion times, that of the structure of a spiral galaxy. At one extreme the observing apparatus may be an electron microscopic or a scanning-tunneling microscope, and at the other, an optical or radio telescope.

With crystals, electron diffraction technology reveals certain symmetries which also manifest themselves at the macroscopic level. Crystallographers identify five possible Bravais or space lattice types in two dimensions, and fourteen types, in three dimensions. All of the two dimensional and some of the three are found in Man's artistic creations, in his art and architecture. A millennium before crystallography became a science, Moorish artists -- Sunni Moslems, forbidden to produce likeness of humans -- were creating magical calligraphy and geometric designs displaying intuitive understanding of the space lattices. This is nowhere more dramatically illustrated than in the stone carvings at the Alhambra Palace in Granada and in the Great Mosque in Cordoba.

Meanwhile, the recurrence of certain numbers and ratios in nature were being incorporated usually unwittingly, but sometimes consciously, by artists into their creations. The numbers embodied in the Fibonacci Series (1, 1, 2, 3, 5, 8, ...) are very often the very same ones significant in genetics and in phyllotaxis (pertaining to arrangements of leaves and branches on plants). The ubiquitous ratio issuing from the series, ($\phi = 1.618\ 034 \ldots$), is approximated in the proportions of the Cheops and Chephron pyramids, in the Parthenon, in the paintings of Leonardo da Vinci, in the architecture of Le Corbusier, in the music of Mozart and Bartok, and in the altogether commonplace -- in three-by-five index cards, and in postcards.

Just as symmetry can produce a sense of harmony, balance and proportion, *too much symmetry* in certain contexts, such as in an endless line of row houses, can have negative emotional impact. And conversely, just asymmetry can produce a sense of discord and lack of proportionality, in some instances, such as in the shape of an egg, (in distinction to a smooth sphere), can generate a positive emotional response -- a sense of release and freedom. Thus, released from the prejudice of viewing only perfect symmetries as ideal, the Alps can be seen as magnificent. Likewise, the finest examples of visual art and music are anything but endlessly regular. Indeed, the notion of "the monotonous" is one of artistic or social aversion. Subtleties in the laws of nature often involve recognition of asymmetries or broken symmetries. Indeed, physical reality melds elements of symmetry and asymmetry. Total symmetry would require absolute and endless homogeneity. Total asymmetry would mean complete chaos, or total absence of order.

It is not the purpose of this study to make an exhaustive inventory of examples, of natural and artistic phenomena which demonstrate the same patterns, but rather to scrutinize the symmetries and patterns at a fundamental level, to analyze the possible forces which might produce similar shapes at wildly disparate scales, to review the notion of aesthetics, and the mathematics underlying aesthetics. By studying the interdynamics of art and science what gains a sense of the confluence of art and science, and to a lesser extent, a modicum of the psychology underlying the human affinity for symmetry. This last message, however, will be regarded as a tacit one, since any serious cogitation on the psychology of art summons forth a picture of fishing in muddy waters.

I. Mathematical Mosaics

> Though God has given to men the best and most perfect understanding of wisdom and mathematics, He has allotted a partial share to some of the unreasoning creatures as well... This instinct is especially marked among bees... They prepare for the reception of the honey the vessels called honeycombs, with cells all equal, similar, and adjacent and hexagonal in form.
>
> *Pappus, AD 4th Century (Thomas 39)*

A figure in two dimensions has two types of symmetry. It has 'line symmetry', if a line can be drawn through it so that each point of one side of the line has a matching point on the opposite side at the same perpendicular distance from the line. It is readily seen that an equilateral triangle possesses three-fold line symmetry; a square, four-fold line symmetry; a regular polygon on n-sides, n-fold line symmetry. Finally, a circle has infinite-fold line symmetry.

A figure has 'point symmetry' if it can be rotated about a point so that it coincides with original position, but specifically excluding the trivial case of rotation by a full turn, or 2π radians. An equilateral triangle can be rotated about a point at its center by $2\pi/3$ radians and by $4\pi/3$ radians in fulfilling the condition above. A square can be rotated through a point at its center by $2\pi/4$, $4\pi/4$ and $6\pi/4$ radians in order to replicate the original picture. A regular polygon of n sides should possess (n-1)-fold point symmetry, with rotations by $2\pi/n$, $4\pi/n$, $6\pi/n$, ..., $2(n-1)\pi/n$ radians all recreating the original position.

The expression 'mathematical mosaics' refers to configurations of regular polygons which completely cover a surface, so that an equal number of polygons of each kind are arranged around a regular array of points called lattice points. Six equilateral triangles can be arranged in this manner and rotations by $2\pi/6$ radians would carry any one triangle onto an adjacent one. This situation could be described as a three-fold symmetry axis existing in a lattice, seen in Figure 1. Similarly a four-fold symmetry axis could exist with squares arranged around a lattice point; there a $\pi/2$ radian rotation would carry a square onto an adjacent one. Finally, a six-fold symmetry axis could exist with regular hexagons arranged around a lattice point, seen in Figure 2.

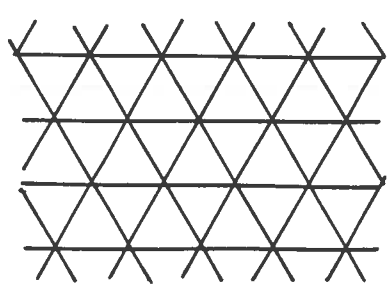


Figure 1. Equilateral triangles, used in tiling a flat surface. Squares can also be used for the same purpose.

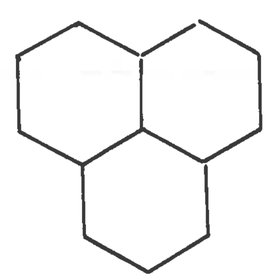


Figure 2. Hexagons, used in covering a flat surface. Indeed, the hexagons can be regarded as the triangles in Fig. 1, taken six at a time.

Here rotations by $\pi/3$ radians map hexagons onto adjacent hexagons. If the surface is to be covered by the same kind of regular polygon, the possibilities turn out to be limited: equilateral triangles, squares, and hexagons.

It is seen in Figures 3 and 4 that pentagons (which call for rotation by $2\pi/5$) or heptagons (by $2\pi/7$) cannot by themselves produce mosaics. It is also impossible to make mosaics with regular octagons or dodecagon only. If one is not restricted to regular polygons, then other shapes, such as rectangles, parallelograms, isosceles triangles can combine in homogeneous arrangements. Finally, if one is not restricted to homogeneity, i.e. a single type of tile, then combinations of different shapes can also be brought together to fill out surfaces in a manner in conformity with the definition of mathematical mosaics given earlier.

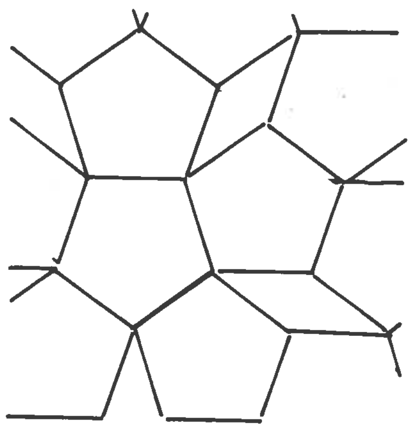


Figure 3. Pentagons arranged contiguously leave gaps in the shape of rhombuses.

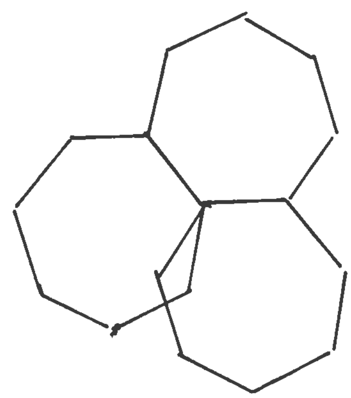


Figure 4. Contiguous Heptagons on a flat surface result in overlapped areas.

In Figure 5, one can see regular polygons of octagons and squares clustered together, and in Fig. 3, pentagons and rhombuses. In fact, the number of kinds of regular polygons on the plane is unlimited.

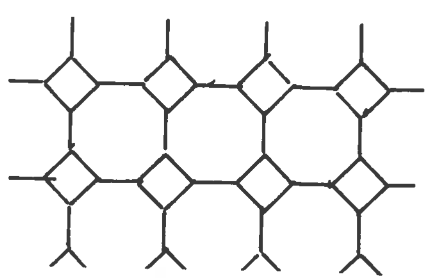


Figure 5. A combination of octagons and squares used in tiling a surface.

It was seen above that among regular polygons, only those with angles at the summit of $\pi/3$, $\pi/4$, a $\pi/6$, (each a submultiple of 2π), or triangles, squares and hexagons would allow equipartition of the plane. In the case of the hexagon, however, the wall-material is minimized. The honeybee, referred to by Pappus, builds the honeycomb with hexagonal cells, thereby minimizing the required wax, and presumably the labor.

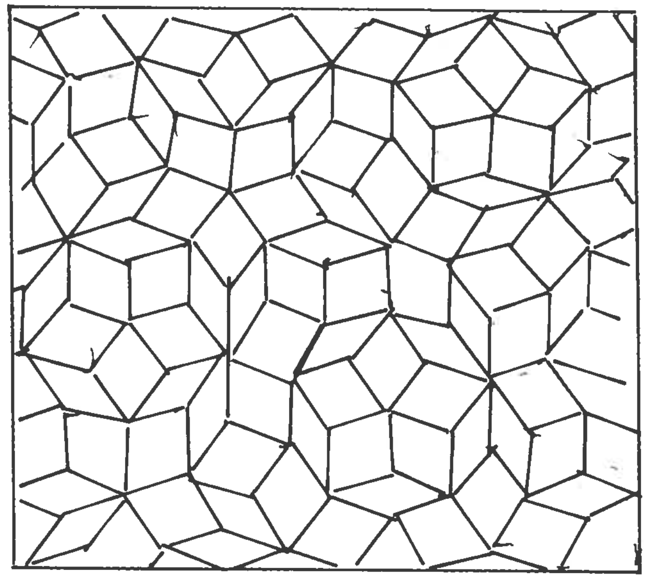


Figure 6. Penrose Tiling, in which rhombuses of two different forms have been used in tiling a flat surface. The ratio of the number of 'fat rhombuses' to that of 'skinny rhombuses' arranged on an infinite plane turns out to be 1.618 034. Moreover, the tiling has the physical significance of being a two-dimensional analogue of a quasi-crystals.

Beyond the regular polygon used in mathematical tiling are a virtually unlimited number of abstract shapes, all understood in terms of symmetry operations allowed on a flat surface. The Moors of Northern Africa (and Spain), and the Selçuk and Ottoman Turks of the Middle East developed two dimensional abstract design and calligraphy to a level of extraordinary sophistication and beauty. The Moors, in particular, adorned their special buildings with patterns revealing tacit understanding of space symmetry concepts, epitomized in the tiles of the Alhambra Palace in Granada, and the Great Mosque in Cordoba. In Figures 7 and 8 are reproduced a pair of examples from the latter location. In the first a leaf motif appears in two shades, a dark leaf pattern situated horizontally, and a light one, vertically. Obvious symmetry operations are translation, as well as vertical or horizontal reflection. In addition, rotation by π radians around a lattice point, also leaves the overall pattern unaffected. If the two different shades are ignored, then rotation by $\pi/2$ radians also leaves the pattern the same. Furthermore, there is the two-fold symmetry in reflection across a horizontal line, and so too, across a vertical line.

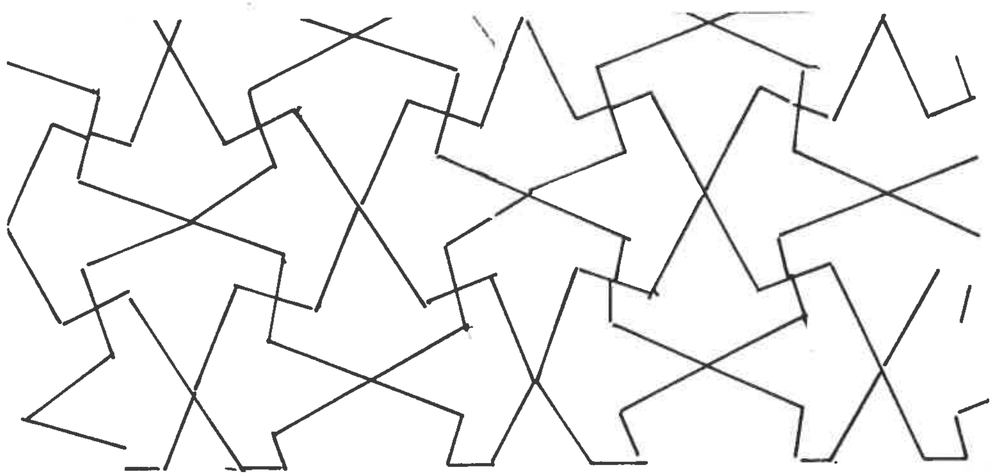


Figure 7. Leaf motifs from the Great Mosque in Cordoba, appearing in two different shades not seen. (Bronowski 73)

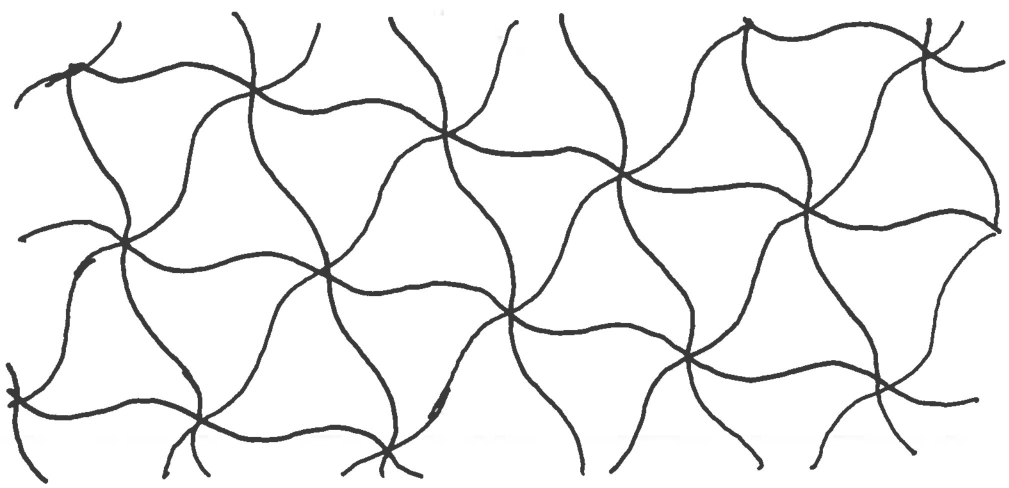


Figure 8. Stylized triangles also from the Great Mosque in Cordoba. The triangles are actually seen in four different shades, not seen in the illustration above.

In Figure 8, four shades are displayed in the pattern of stylized equilateral triangles. The pattern is invariant to translation vertically and horizontally. A subtle quality of these symmetric figures, however, leads to violation of reflection symmetry. If these triangles can be regarded as 'clockwise-twisted', then reflection across horizontal or vertical lines would render them 'counterclockwise-twisted.' If the color differences are ignored, the pattern also possesses six-fold rotational symmetry against rotation by $\pi/6$ radians, evocative of Figure 1 for equilateral triangles.

The Twentieth Century Dutch artist M. C. Escher employed symmetry operations in order to generate inexhaustible patterns of realistic figures in his graphic artwork, several of which are reproduced in Figures 9-12. Unencumbered by religious interdicts such as the ones imposed on Islamic artists, Escher drew human and animal figures as often as he did abstract design.

In Figure 9 and 10 the patterns represent homogeneous figures of horsemen in one case, and swans in the other. There exists the obvious symmetry in translation in vertical and horizontal directions. However, because of the two-color design, reflection (plus translation diagonally) does not leave the pattern invariant. Ignoring the differences in colors, this symmetry would be secured.

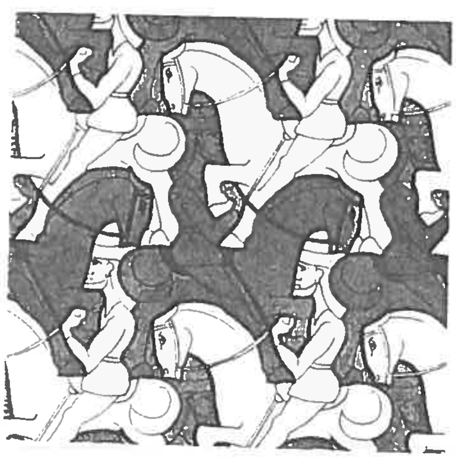


Figure 9. Horsemen. (M. C. Escher)

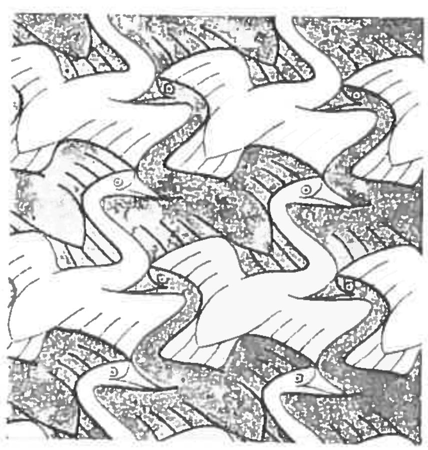


Figure 10. Swans.

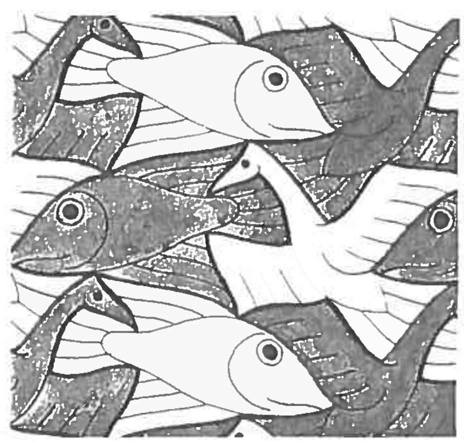


Figure 11. Fish and Birds.

In Figure 11, a mosaic has been created from two different types of figures, and in two colors. Again, translation vertically or horizontally leaves the pattern unchanged. Reflection symmetry would be established only if the difference in color is suppressed.

In Figure 12, entitled 'Drawing Hands', the symmetry operations are much more difficult to analyze in terms of translation, rotation and reflection. Nevertheless, as a realistic rendering of a pair of hands seen at different angles, it depicts a fascinating study of a right hand giving actuality to a left hand, which is, in turn, creating the right.

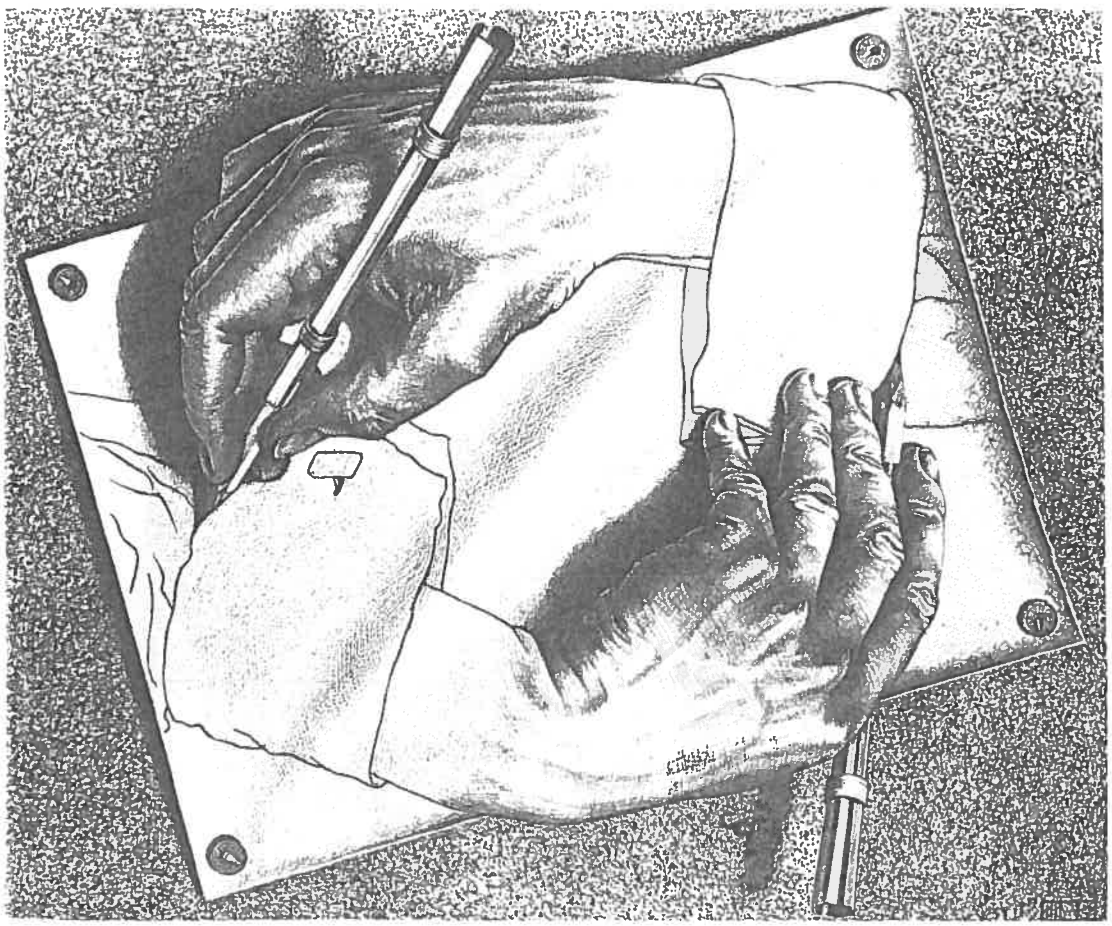


Figure 12. 'Drawing Hands'. At first pale, Escher's illustration appears to represent the image of a hand reflected in a mirror, but clearly bi-lateral symmetry is absent. Further scrutiny, suggests point symmetry -- or a two-fold rotational symmetry, but neither is that the case.

Polyhedra

In the last section it was seen that the different types of regular polygons which could create a homogeneous mosaic were only three in number. However, irregular mosaics, consisting of different regular polygons, are unlimited in number.

'Regular polyhedra' are defined as three dimensional shapes, comprised of regular polygons for their surfaces, with all the surfaces, edges and vertices identical. They are also known as 'platonic solids'.†
The five types of regular polyhedra are the tetrahedron, the octahedron, the icosahedron, the cube, and the dodecahedron, can all be seen in Figure 12.

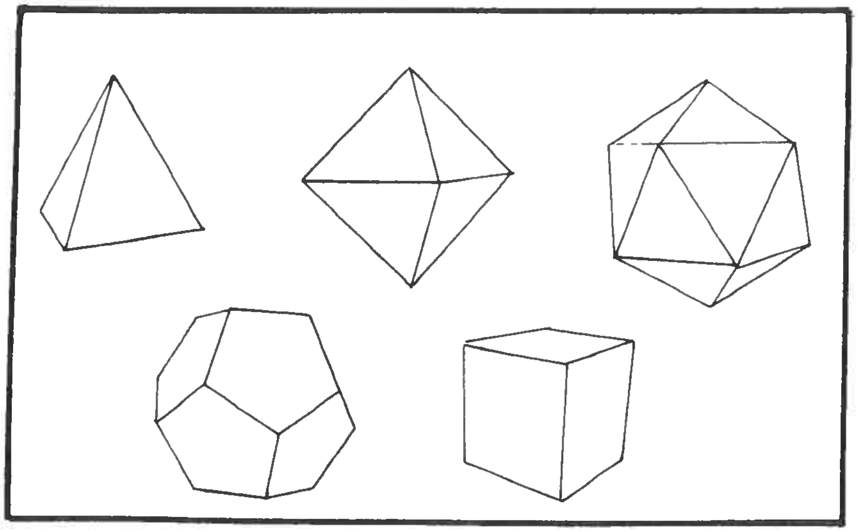


Figure 12. The Regular Polyhedra.

- The tetrahedron is comprised of four equilateral triangles, with four vertices, and three triangles at each vertex.
- The octahedron consists of eight equilateral triangles, six vertices, with four triangles at each vertex. (Let us reiterate that six equilateral triangles around a point would lie flat.)
- The icosahedron consists of twenty equilateral triangles, with twelve vertices, and five triangles at each vertex.
- The cube consists of six squares of eight vertices and three squares at each vertex. (Recall that four squares around a point would have formed a flat surface.)
- Finally, it will be recalled from Figure 3 that three pentagons arranged around a point failed to close up on the plane. If the gap is to be closed, a 'cupping' occurs, and the vertex thus created becomes one of the twenty vertices of a dodecahedron, a three dimensional figure which has twelve pentagons for its surface.

By mixing a variety of regular polygons, while adhering to the condition that at all vertices the arrangement of polygons is the same, one obtains the closed shapes called 'semi-regular polyhedra'. For

† For the Ancient Greeks there existed only four elements in nature--earth, fire, air and water--and different admixtures of these elements explained the composition of all materials in nature. To Plato atoms of fire had tetragonal shape; atoms of earth, cubic; atoms of air, octahedronal shape; and those of water, icosahedronal.

example, the icosahedron with twenty equilateral triangles, in having its vertices cut, becomes a 'truncated icosahedron', a figure characterized by two hexagons and one pentagon at each vertex. Another semi-regular shape the cuboctahedron with fourteen faces, (eight of which are equilateral triangles and six squares), is created by taking the centers of the edges of a cube or of an octahedron. Here a physical significance is to be found: the arrangement offers the model for close-packing of identical spheres in space, which is of considerable interest in crystallography. In the following section the classification of crystal structure will be presented within a more general topic about patterns in nature.

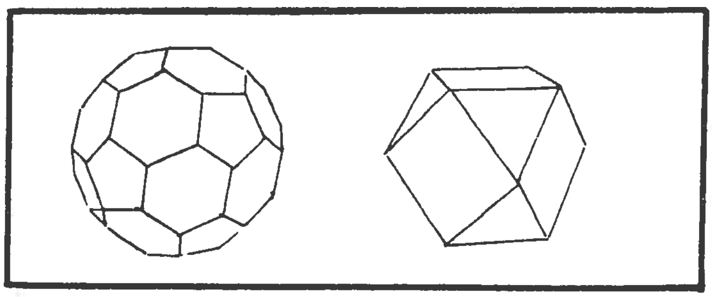


Figure 13. Two Semi-Regular Polyhedra: The Truncated Icosahedron and the cuboctahedron.

II. PATTERNS IN NATURE

At a fundamental level, the symmetry of atomic orbitals determines the type of bonding in atoms, and ultimately the properties of matter in the bulk, such as hardness, conductivity, boiling point, transparency, etc. At a still more fundamental level, it is the symmetries in nucleon orbitals which is intimately connected to nuclear shapes, nuclear stability, and ultimately to the relative abundance of elements in nature. Serious discussions of orbitals educes quantum mechanics and its exclusive domain of wavefunctions, and would take us too far afield. The shapes of macroscopic crystals, including gemstones, however, involves geometric arrangement of atoms at the microscopic scale, and can be discussed in terms of space symmetry operations without eliciting quantum mechanics.

The Architecture of Crystals and Quasicrystals

There are only certain kinds of symmetries which our space can support, and it is these that are enshrined in crystals--which are after all well ordered arrangements of identical unit cells, fitting together regularly and periodically to fill space. Mathematical mosaics in two dimensions become a good analogue for the structure of crystals in three dimensions. It was seen in Section II that the three-fold, four-fold and six-fold rotational symmetries corresponding to the equilateral triangle, the square and the hexagon. And five-fold symmetry was specifically ruled out since the associated pentagons alone could not tile a plane.

In three dimensions, although a single molecule can have any degree of rotational symmetry, an infinite periodic lattice cannot. Further, it is possible to make a crystal from molecules which individually have five-fold rotation axis, but the lattice cannot support a five-fold rotation axis. The unit cell structure of most crystals is based on Platonic solids, such as the cube, tetrahedron and the octahedron.

In 1984 an entirely new class of materials were first identified which appear to have a structure based on another type of platonic solid, the icosahedron, which has equilateral triangles for its twenty faces. However, since five faces meet at each vertex, there is a five-fold rotational symmetry, in violation of one of the most fundamental theorems of crystallography. This class, which Nelson (86) calls 'Schectmanite' is represented by certain materials, including an alloy of aluminum-magnesium, which are cooled extremely rapidly.

The structure of quasicrystals can best be understood in analogy with Penrose Tilings, in which a plane is tiled aperiodically, but possesses long range translational order, as well as orientational order (recall Figure 6). Consisting of only two shapes, both of which are rhombuses--but one with internal angles of 36 and 144 , and the other with 72 and 108 , the tiling calls for certain 'matching rules'. Like quasicrystals, Penrose Tilings have a kind of five-fold symmetry, since the parallel lines of the two types of rhombuses intersect at angles which are multiples of 72 , or one-fifth of a circle. Of particular interest for the present paper, the ratio between the number of 'fat rhombuses' and 'thin rhombuses' is the 'golden mean', (1.618 034), the subject of Section IV.

Spirals in Nature

The geometric regularities of gemstones are manifestations of the same regularities found at the microscopic scale, and space symmetry operations and mathematical tiling rules assist in classifying the different types of crystal structures. An entirely different pattern in nature, the spiral or coil, is encountered at dramatically disparate scales, and in basically three different forms--the hyperbolic, the Archimedian, and the logarithmic. The rather obvious key to spiral formation of any type is the different growth rates of two surfaces, with the slower growing surface being gradually enclosed by the faster growing one. This simple rule offers a connection among strikingly different phenomena, and it is only when one takes a moment to ponder, that the commonplace becomes wondrous, and the wondrous, commonplace: Among others, one might contemplate the shapes of the chambered nautilus (seen in Figure 15); the tusks of a mastadon; pairs of horns on a ram; shavings from a wood plane; the claws of a cat; the dried leaf of a poinsettia; the fangs of a saber tooth tiger; the myriad of gastropods; the human lip, curved gently outward, with the inner tissue growing faster than the outer; the swirling cloud patterns in a hurricane; the arms of a spiral galaxy which can stretch hundreds of thousands of light years across (seen in Figure 19). Finally, there is the agent of the genetic code, deoxyribonucleic acid or DNA. Here a pair of columns of sugar and phosphate molecules assume their characteristic spiral shape because of the unequal lengths in their edge bond. In short, it is skewed molecular units which give shape to the double helix. Moreover, the actual proportion exhibited by the DNA, according to Harel et al (86), is 1.62, (Figure 16).

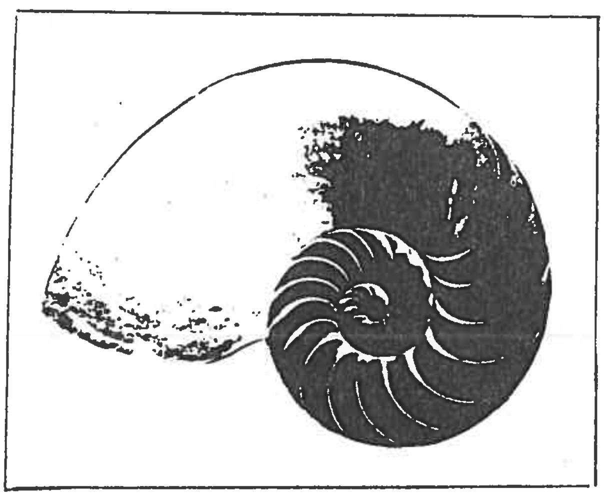


Figure 15. The Chambered Nautilus

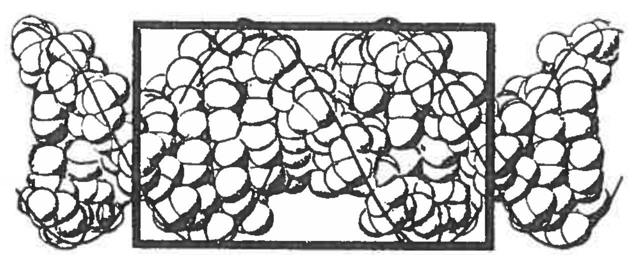


Figure 16. The DNA Molecule with length-to-width ratio within one cycle of 1.62. See Harel et al (86)

The hyperbolic spiral is described in polar coordinates by $r=-k/(\pi/2+\theta)$ with k constant, (Figure 17). The curve becomes asymptotic to $x=-k$ in the limit when $\theta \rightarrow \pi/2$. Fronds of the sago palm and the fiddlehead, a type of fern, resemble hyperbolic spirals in that they leave the stem straight and only at their tips become coiled.

The Archimedian spiral, which is also observed in nature, appears similar to a strip of uniform thickness, coiled tightly around a central axis. It is a spiral which replicates the grooves of the old-fashioned record, or a roll of tape. Magnified 110,000 times, the cross section of the axonome of the giant heliozoan, E nucleofilum, displays a microtubule pattern of a pair of Archimedian spirals, each wrapped about five full turns around a common axis . (See Figure 18. The tight coiling spiral is attributed to the short-ranged immediate neighbor interactions of microtubules. (Roth and Pihlaja, 77). As pointed out by Stevens (74), the Archimedian is displayed by the primitive sea slug Dictyodora, which evolved into a creature resembling a corkscrew. In this instance the spiral has been stretched out along its axis. Paleontologists explain that the creature's evolution in a uniformly wound spiral assured the most thorough foraging of well defined areas, the distant ancestor of the Dictyodora, it seems, was not in coiled form, and use to wander around in haphazard manner. The two processes are rather evocative of the undersea search for fragments of the shuttle *the Challenger* in 1986.

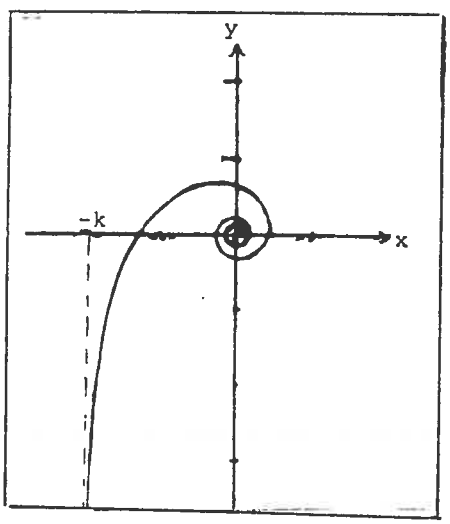


Figure 17. The Hyperbolic Spiral.

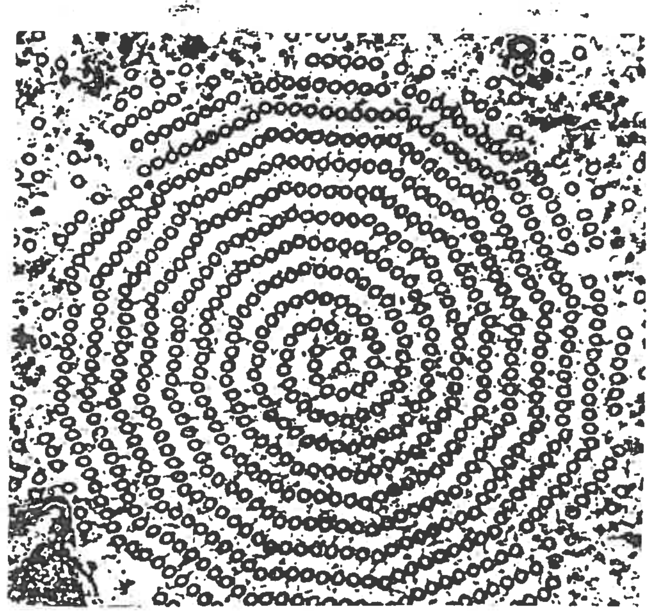


Figure 18. The cross section of axonome of the heliozoan, x110,000. (Roth and Pihlaja, 77)

The linear dependence of the radius on the angle for the Archimedian spiral can be characterized by $r = a\,\theta + b$, expressed in polar coordinates, with a and b constants.

The most significant of the spirals, the logarithmic, is described by the expression (in polar coordinates)

$$r = r_o e^{\beta\theta} \qquad (1)$$

where r_o and β are constants. The logarithmic spiral finds its most dramatic example in the shell of the chambered nautilus. (Recall Figure 15.) Called a 'living fossil' because of its existence as far back as 200 million years ago, it has been found to be anything but an inert creature, having evolved continuously and rapidly during its existence. The creature inhabiting its domain, the shell, extends and enlarges it while forming a continuous rolled tube. As Stevens (74) points out, the difference in growth in the successive chambers automatically causes the coiling to take place, and that no gene need to remember or plan the final shape of the shell. Rather, it needs only to facilitate a difference in growth between inner and outer surfaces of the shell.

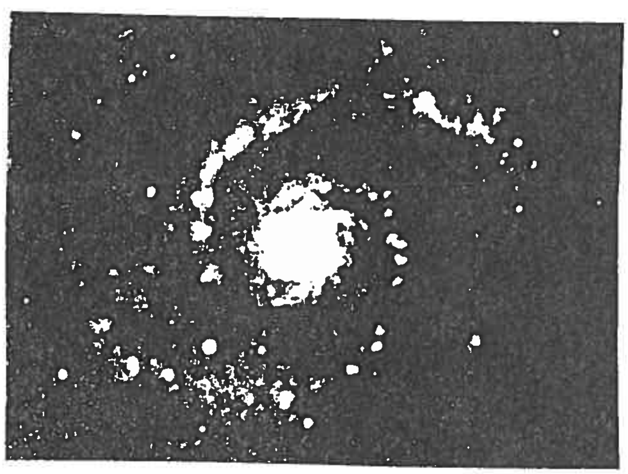


Figure 19. Galaxy M51 , 'Whirlpool'.

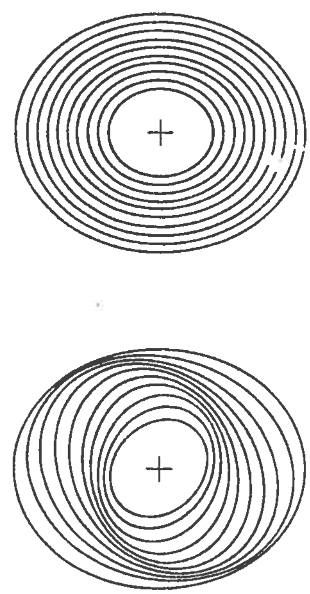


Figure 20. Superimposition of elliptical orbits giving rise to spiral patterns.

Logarithmic spirals are again encountered in spiral galaxies. In Figure 19 is seen the galaxy M51, otherwise known as the 'Whirlpool Galaxy'. A pair of spirals are clearly identifiable. In Figure 20 the model by Kalnijs (Kaufmann 79) for the formation of spiral galaxies is seem in a pair of schematic pictures which serve to illuminate the conditions for the birth of spiral galaxies. According to the work of Kalnijs, the initially independent concentric orbits of stars are forced into an overall correlated pattern by spiral density waves, presumably initiated by perturbations. The energy fueling the density waves is thought to be generated deep within the central regions of the galaxy. The latter conjecture is not entirely well understood, just as questions regarding whether, (or why not?), the arms of a spiral galaxy disappear by becoming wrapped up, as one would expect from the differential rotation in the body of the galaxy.

In this rather far ranging discussion on spirals in nature, an area which deserves special attention is the one of spirals in plants. The arrangements of petals on flowers, of leaves on stems and of branches on trees comprise an area of botany known as 'phyllotaxis', where one has long recognized the curious appearance of certain numbers such as 2, 3, 5, 8, etc. (or Fibonacci numbers). The intervals between the thorns on a rosebush, the intervals (measured in arcs) between branches in the poplar tree, and those in the delicate network of veins in the leaves of the hardy and rigorous ivy plant are disparate examples of plants which utilize these numbers. While the proportions enhance the beauty of the plant, the geometry is found to be functional--enabling the plant to obtain for its parts maximum exposure to the sun, or maximum nutrients for its cells, etc. The Fibonacci numbers will be discussed in Section IV. Meanwhile, the remainder of this section will be devoted to examining spiral phyllotaxis.

Spiral Phyllotaxis

Stevens (74), who investigated numerous plant patterns, ranging from branches of trees to delicate petals of the most ephemeral flower, describes the cross section of the celery just above the meristem (the conical mound of solid tissue at the base of the plant). The stalks of the celery are packed together closely, creating a swirling pattern. A cursory glance reveals a pair of clockwise spiral patterns, countered by one counterclockwise spiral. A closer scrutiny, however, reveals a more general recipe: When leaf bases develop in succession around the stem apex, they fit between each other in a manner which composes a helical pattern with each stalk riding above the older member of a pair of stalks in the preceding whorl; and variations of the helical pattern appear with stalks of one whorl interpenetrating and making contact with stalks of the previous whorls.

If one examines the helix of thorns of a young hawthorn tree, one finds in two full circles around the stem a total of five thorns, an arrangement characterized by 2/5. Apple, oak, and apricot have the same 2/5 pattern of phyllotaxis. As for sedges, beech, and hazel, a phyllotaxis of 1/3 is found; for plantain, popular, and pear, 3/8; for leeks, willow, and almonds, 5/13. These all exhibit helical spirals.

When the helix is compressed into a plane, compound spirals, such as in the celery's phyllotaxis can appear. Other compound spirals are the pineapple, 8/13; daisies display clockwise to counterclockwise spirals of 21/34. Sunflowers, depending on their size, can have phyllotaxis of 21/34, 55/89 or, in the case of a giant sunflower, 144/233. In each instance, the numbers in the numerator and those in the denominator are consecutive terms in the Fibonnacci series.

The reason for the occurrence of the Fibonnacci numbers in such a diversity of plants turns out to be a necessary consequence of the growth pattern inherent in all of them. This is demonstrated by an analysis offered by Stevens, who begins by plotting the tips of a bunch of celery stalks. Through the points it is possible to draw a continuous logarithmic spiral. The circular arc between any two consecutive points is found in a precise measurement to be 137° 30'28" (=137.5077°). This value, compared with a full circular turn of 360° is

$$\frac{137.5077^\circ}{360^\circ} = 0.381\,966,$$

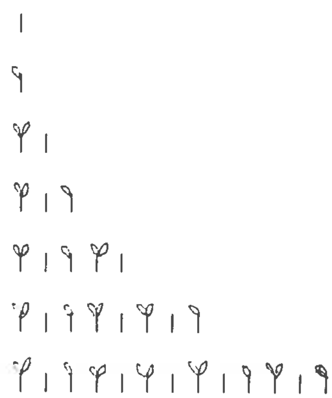
a value equal to the square of the inverse of 1.618 034. Thus, the spiral's relation to the Golden Mean emerges. More significantly, Stevens draws all the possible smooth spirals through these points, both clockwise and counterclockwise. What issues is quite dramatic: Compound spirals with only phyllotaxis of 1/2, 2/3, 3/5, 5/8, 8/13, 13/21. The array of points not only generates all Fibonacci fractions, it generates *only* Fibonacci fractions.

One must, however, consider these patterns in the proper perspective. The plant is no more enamored of the Golden Mean than it engages in mathematical computation before sprouting a stalk. Rather, it puts the stalks where they have the most room, where they can make the most of the nutrients and the sunlight available to the plant. As Steven's observes, "All the beauty and all the mathematics are the natural by-products of a simple system of growth interacting with its spatial environment."

III. THE FIBONACCI SERIES AND THE GOLDEN MEAN

Leonardo Fibonacci in his book *Liber Abacis* (AD 1202), introduced a pair of seminal thesis: The first was meant to elucidate the merits of the decimal system† ; the second, to discuss the propagation of a pair of rabbits left in an enclosure. It is the latter which gives rise to the series which bears his name, as well as to the irrational number 1.618... known variously as the 'golden mean', the 'golden section', the 'golden ratio', and hereafter denoted by ϕ.

In Fibonacci's problem of the rabbits, the rules are spelled out as follows: (i) A mature pair of rabbits can give rise to one new pair per month; (ii) The offspring will have to mature for two months before they can begin to reproduce themselves; and (iii) No new rabbits can be introduced from outside, and no rabbits can leave the enclosure. In order to visualize the propagation of the rabbit population we introduce the symbols "  ", for a mature pair capable of reproducing; "  " a pair which is only a month old and not capable of reproducing, and finally, " | ", a brand new pair of offspring. The very first pair of rabbits in the series is an immature pair. The numbers of pairs in successive months can be seen in the following table.



Month	Number of Pairs
1st	1
2nd	1
3rd	2
4th	3
5th	5
6th	8
7th	13

For example, in the 3rd month there should be two pairs of rabbits: the original pair (), now fully matured, plus one new pair (|). (Of course, with one ear, the rabbits cannot reproduce--*as it is all in the ears!*).

---

† The Arabs had brought the decimal system from India around AD 750. However, it was not until the Thirteenth Century, in large part with Fibonacci's efforts, that the system began to take hold in Europe. The symbols known as *Arabic numerals*, 0, 1, 2, 3, ... , were adaptated by Fibonacci from the symbols used by the Arabs, ٠, ١, ٢, ٣,...

The formal Fibonacci Series has as its basis the three statements:

$$u_1, u_2, u_3 \ldots u_n \qquad (2)$$

$$u_n = u_{n-1} + u_{n-2} \qquad (3)$$

$$u_1 = u_2 = 1, \qquad (4)$$

with n=1, 2, 3, ...

Thus 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, ... Defining $R_n = u_{n+1}/u_n$, one finds for R_1, R_2, R_3, ...

1, 2, 1.5, 1.67, 1.60, 1.63, 1.615, ...

converging in the limit:

$$\lim_{n \to \infty} R_n = \frac{1+\sqrt{5}}{2} \qquad (5)$$

approximately, 1.618 034. A pair of intriguing features is that the square of ϕ is 2.618 034..., and the inverse, 0.618 034 ...

The terms of the series can be computed with the recursion relation (3). Moreover, it can be shown (see Appendix B), that the nth term can be computed directly from

$$u_n = \frac{(1.618\,034\ldots)^n - (-0.618\,034\ldots)^n}{\sqrt{5}} \qquad (6)$$

In the following section some of the geometric manifestations of the Fibonacci Series will be reviewed.

Geometric Constructions Associated with the Golden Section

Starting with the square (ABCD) with sides of unit length, one can construct the 'golden rectangle' quite simply by bisecting the square, then by using the diagonal MC as the radius of an arc CF (Figure 21). Next, one extends AD horizontally to intersect the arc CF, draws a perpendicular at this intersection, and finally extends BC to complete the rectangle. Since DC has a length of unity, and MD of 0.5, the Pythagorean Theorem yields MC as $\sqrt{1.25}$ or approximately 1.118 034. Adding AM = 0.5 to MF = MC, one immediately arrives at 1.618 034. The rectangle ABEF is then the Golden Rectangle.

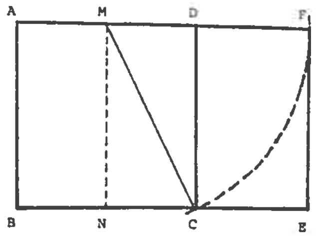


Figure 21. Construction of the Golden Rectangle.

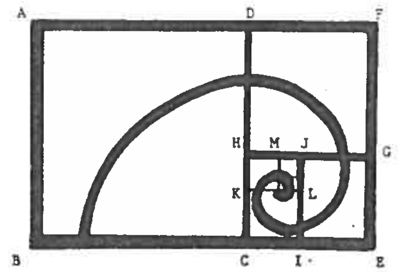


Figure 22. 'Whirling Squares' and the Logarithmic Spiral.

Moreover, DF = MF - MD = 1.118 034 - 0.5 = 0.618 034. The ratio FE/DF is equal to 1/0.618 034 = 1.618 034, which renders the rectangle DCEF a golden rectangle also. Within DCEF apportioning off the square DFGH creates another golden rectangle yet in HGEC, Figure 22. The process can be repeated *ad infinitum*, each time creating a square plus a new golden rectangle with an extant golden rectangle. Finally, if one connects the centers of the squares, with a continuous curve, one obtains the logarithmic spiral, and the apt description 'whirling squares'.

In Figure 23 is plotted the logarithmic spiral represented by

$$r = r_0 \exp\left[-(\ln \phi/\pi/2)\theta\right]$$

which is just equation (1) with $\beta = -\ln \phi / (\pi/2)$. The intersections of the spiral with the -x, +y and +x axes is seen to produce two sides of a golden rectangle. In Figure 24 radial lines have been drawn intersecting the spiral at equal angles. The resulting figure represents a remarkably accurate cross section of the chambered nautilus, seen earlier in Figure 15.

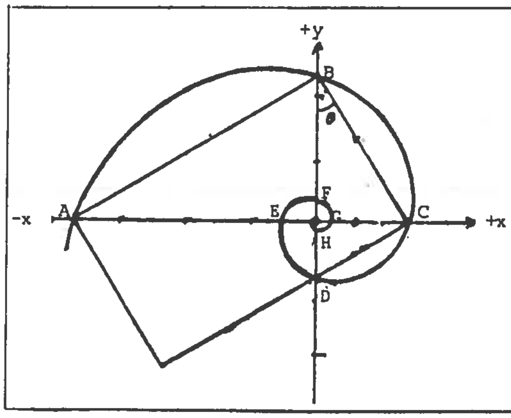


Figure 23. The logarithmic spiral plotted from equation (1).

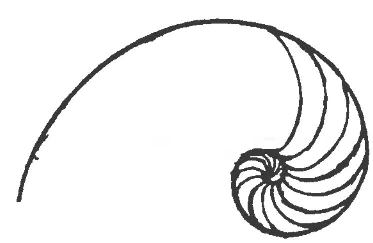


Figure 24. The logarithmic spiral, with equiangular partitions.

Finally, in one last construction (Figure 25) the logarithmic spiral is generated from a 'golden triangle' of angles 36° , 72° and 72° . For such a triangle AB/BC = ϕ . Furthermore, the points D, E, F, G. ... used in generating the spiral all give segments related to the golden mean, AB/BC = BC/CD = CD/DE = DE/EF = ... = ϕ .

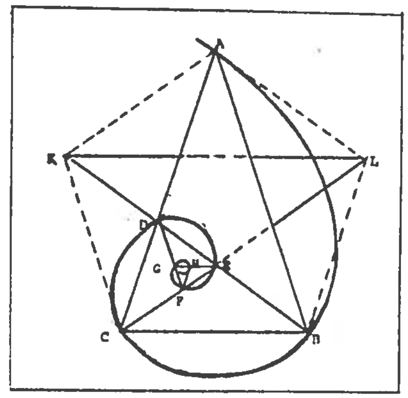


Figure 25. The 'golden triangle' generating a logarithmic spiral and the pentagram.

The triangle itself is significant for its use in certain Renaissance paintings, a point which will be discussed in the next section. More immediately, the 36° vertex is seen as one prong of a five-point star, the pentagram, which in turn, yields the pentagon when the points are connected. The latter figure was known as the 'magic penticle' to the Pythagoreans of Ancient Greece, and later it became a favorite conjuring device for magicians. In this capacity it endured for centuries.