Committee 2
Symmetry In Its Various Aspects:
Search for Order in the Universe

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SYMMETRY IN THE LAWS OF PHYSICS

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SYMMETRY IN THE LAWS OF PHYSICS

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Abstract

Invariance considerations were introduced in Physics by Galileo, Euler, Lagrange, Hamilton and Jacobi. Emmy Noether's 1918 formulation best abstracts their dominant role in Modern Physics, where Einstein ushered in symmetry, first in kinematics (1905) and later in dynamics (1915). The same sequence recurred in the "Standard Model", the ruling paradigm of Quantum Physics (1974), where the dynamical "Gauge Symmetries" are superimposed upon the basically kinematical SU(3) "flavor" classification (1961). A surprising feature is the overall geometrization, imposed by phenomenology rather than choice, realizing in Physics the Erlangen Program, as launced in Mathematics by Klein and Lie in 1872.

1 The Concept of Symmetry

Symmetry is the name given to a situation in which the evolution of a physical system remains unmodified, even though we do modify the values of some variables. These modifications can be *active*, namely an actual motion relative to a given frame of reference (either in spacetime or in some configuration

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space) - or ***passive***, i.e. it is the frame of reference which undergoes the modification, thus bringing about a relabelling of the system's parameters. The problem of the ***hydrogen atom***, for instance, has *spherical symmetry*: there is just one particle - a proton - in the nucleus, which can thus be assumed to be spherical, and the magnitude of the electrical (Coulomb) potential - in which the orbiting electron finds itself - depends only on the distance from that central proton, not on the direction (in polar coordinates, $V(r)$ rather than $V(r, \phi, \theta)$). The nondependence on θ, ϕ is a statement of invariance, since we can vary the values of these angles without affecting the system's dynamical evolution. Another point of view with respect to the passive mode is due to E. Whittaker [1], who dubbed such invariances *Postulates of Impotence*, since one cannot select a *preferred* reference frame.

It is natural that the concept of symmetry should dominate physics. Indeed, the scientific method is based on the assumption that there are *laws*, i.e. rules that have to be obeyed by material bodies, with no "preferences". If A and B are equal before the law, then the physical evolution will be the same if we replace A by B. Note that saying that they are "equal" before a given law means that the relevant variables are the same: for Coulomb's law, for instance, this would imply that A and B carry equal electrical charges and are at the same distance from a "central" charge C. [Note that in *classical* physics, this was always true, but in *quantum* physics it is not always so. Certain particles, obeying (so-called) "Fermi statistics", display "individuality", so that no measurable feature can be said to be "irrelevant". Exchanging A and B, in these cases, does entail physical consequences]..

Sometime, although the dynamical evolution is not unaffected by the modification of certain variables, there is some feature which does remain unaffected. Such a partial symmetry is termed a *regularity*. In the phenomenological exploration of a new domain, the discovery of regularities is the basic stage. These regularities may then provide clues to the structural and dynamical paradigms of the new domain - and certainly provide a way of testing them, since the paradigms have first and foremost to reproduce the regularities. The first clear such example in post-medieval physics is provided by Kepler's laws: all three laws make statements about features that remain unaffected under active transformations of the physical variables. In the first law, the identification of the elliptical shape of the planetary orbits implies the invariance of the summed distances from the two foci (the ellipse's definition as a *locus*) even though we modify the planet's position along the orbit; the second law is an explicit such invariance statement, for the areas swept by the radius-vestor to the planet during equal time intervals, at different points along its trajectory; the third law does the same for a certain ratio, upon replacement of the planet by another (resulting in a different orbit), i.e. transforming the mass, position and velocity.

Mathematically, symmetries and regularities involve *algebraic* considerations. What is required is an evaluation of the effects of transformations (*discrete or continuous*) on a set of variables; this is precisely the content of the theory of *transformation groups*. In his preface to the second edition of his book *Group Theory and Quantum Mechanics*, Herman Weyl condoles [2] with the disappointed physicists, who were hoping that "the group pest" would soon be over. Weyl explains why some groups (such as the Rotations or the Lorentz group) are there to stay. Yet even Weyl did not dare imagine the extent to which Group Theory would come to dominate the field by the end of the Century [3]. Physics as a user is continuously having recourse to the great classification theorems - Cartan's of the Simple Lie Algebras and Groups [4], that of the Simple Finite Groups (the end result of a weighty collective effort, published in the Seventies [5]) and that of the Simple Supergroups (achieved by V. Kac in 1975 [6]) - and picking from them appropriate answers. As a matter of fact, physics is making use of additional algebraic tools, in categories that have not yet been catalogued and classified - Quantum Groups [7], infinite structures such as *diffeomorphisms*[8], or the Kac-Moody algebras [9], etc.

2 Emergence of the Action Principle

Newton wrote down laws of physics which could provide a full description of the phenomena he was studying (including the reproduction of Kepler's regularities). He did not involve himself in derivations from some "deeper" *first principles.* In addition, although interested in scripture and religion (devoting much of his time to scholastic or cabalistic studies, in his later years), he sems to have taken care not to let the latter impinge upon his scientific work. Some of Leibniz' philosophical ideas, on the other hand, appear to have had some impact on the development of physics - especially in this context of symmetry, which interests us here. Leibniz, like Bishop Berkeley, did not believe in absolute space or absolute time; he stressed *relativity*, i.e. the unphysical standing of the frames of reference, i.e. *symmetry in the passive mode.* [Note that he had however nothing to offer as a replacement for what Newton managed to derive, using absolute space and time. It all had to wait for Einstein. Some ideas cannot help, even if they are correct, until "their time has come"]. Leibniz' other surprisingly fruitful idea - much ridiculed as a philosophy, witness Voltaire's "Candide" - was that "this is the best of all possible worlds". Indeed, Dr Pangloss would rightly be regarded as extremely naive, nowadays. However, seen from the point of view of methodology, the idea is fruitful indeed. Remember that Leibniz' version of the Calculus emphasized its application to the finding of a function's *extrema* - witness the title he chose for his book (1684) on the Calculus: "Nova Methodus pro Maximis et Minimis". Mathematically, the phrase "the best of all possible worlds" implies the existence of some "goodness" function, which could be evaluated for any possible world, with the present one scoring high and representing a maximum of the function.

This was the philosophy behind the development of the Calculus of Variations by the brothers Jakob I and Johann I Bernoulli, with the latter being particularly influenced by Leibniz. The program achieved its useful form of a "Principle of Least Action" in the further work of Euler, Maupertuis and Lagrange (not the "best" of worlds, just the "coziest" - least action..) around 1750. In the XIXth Century, Hamilton and Jacobi presented an alternative formulation. Mathematically, the *action function* I was given by an integral over a path, which should not change (i.e. a vanishing derivative) under some parametric modifications of that path. The integrand is the Lagrangian function $L = T - V$, T the kinetic energy and V the potential energy. This is equivalent to evaluating the variation of the action function and putting it to zero $\delta I = 0$. Although the ingredients (the variations of the canonical variables and of their time-derivatives) are algebraic, the final "cooking" is done in the calculus (of variations..).

Note, however, that from Leibniz to Jacobi, the developers of the Principle of Least Action were not aware of the algebraic aspects of their creation, due to the relative lack of sophistication of the variables, in Classical Mechanics. Had they been dealing with the nuclear interactions, for instance, they would have realized what they lacked and might even have invented Group Theory, just as Newton invented the Calculus. Instead, it was done in 1825-30 by two very young men, a Paris highschool pupil, Evariste Galois and a Norwegian youth, Niels Henrik Abel, both of whose papers kept being mislaid (or thrown away?) by the greatest mathematicians of the age, apparently blind to the potentialities of this emerging branch of mathematics.

The advent of Quantum Mechanics further extended the applications of the Action Principle. Schroedinger's wave-function is closely related to the action and Feynman's *path integral* version of the Quantum formalism is inspired by the Principle of Least Action. Dirac found an elegant adaptation of the Hamiltonian formalism to the Quantum domain.

Up to this point we have not had occasion to mention *geometry*. It enters the story with the 1872 launching of the Erlangen Program [10] by Felix Klein, geometrician, in the presence of Sophus Lie, algebraist. Together, they had conceived this program in 1869, as students in Paris, after listening to

Camille Jordan expositing Galois' work and to Gaston Darboux' presentation of Gauss-Riemann differential geometry. The Erlangen Program suggested using symmetry as a guide to geometry and applying their combined strength as a tool in the interpretation of mathematics in general.

3 Relativity: Kinematical and Dynamical Symmetries

We now come to the XXth Century. It was ushered in by Lord Kelvin's two *dark clouds* [11], namely the null result of the Michelson Morley experiment and the inadequacies of the thermodynamical *equipartition* theorem in explaining the specific heats of solids. Einstein and Planck, respectively, cleared these clouds, thereby launching *Special Relativity* and the "old" *Quantum Theory.* Whether or not Einstein was aware of the Michelson-Morley experiments' paradoxical results, he was realizing and focusing on the inadequacy of the Galilean invariance of Newtonian Mechanics, in dealing with Maxwell's Electrodynamics. Lorentz and Poincare had also encountered the problem but did not cross the "Aetherial" Rubicon. There were two conceptual obstacles which deterred them: the *aether* (with its peculiar solid-like properties) was necessary for the propagation of light and electromagnetic radiation (all transverse), and tinkering with *time* countered one's inborn intuition. Though he was not a mathematician and knew no Group Theory, Einstein realized the requirements of consistency and came up with *a theory whose almost entire content consists in a symmetry requirement, that of the Loremtz or Poincare groups* (which were missed by both Lorentz and Poincare..) Einstein's June 1905 Relativity paper [12a] does away with the aether and constrains the physical vacuum by the requirement of symmetry under these groups (equivalent to the more common statement of the *invariance of the velocity of light*) - with all the non-intuitive consequences, especially about *the relativity of simultaneity.* I have conjectured elsewhere that in doing away with the aether, Einstein was encouraged by his very recent results on the photo-electric effect (March 1905) based on his having physically extended Planck's idea, *treating light as a localized particle* and thus freeing himself from the Maxwellian *wave* aspect - which had brought in the aether, originally. We know from Einstein's later stubborn struggle against Bohr's abstract application of Quantum Mechanics . how reluctant he was to forego an intuitively meaningful picture. I believe he would have hesitated to remove a feature which was supposed to be essential to the propagation of radiation - had he not been under the impact of the new quantum particle-like approach to light.

The Special Theory of Relativity is thus the first major physical theory which is built around a symmetry principle. This became more clear when Einstein's former teacher at the Zurich ETH, the geometrician Hermann Minkowski, in his speech to the LXXXth Congress of the "German Society of Nature Researchers (=Scientists) and Physicians" (Cologne, 1908), explained that Einstein's discovery simply meant that the local geometry of space and time, rather than being Galilean (Euclidean 3-space x one-dimensional time) was that of a *pseudo-Euclidean 4-manifold* with a pseudo-Pythagorean metric,

$$\begin{aligned}(\text{“}Interval\text{”})^2 &= (length)^2 + (width)^2 + (height)^2 \\ &- (time\ duration \times velocity\ of\ light)^2\end{aligned}$$

where time is measured in units of length, by evaluating the distance that would be travelled by light during this time. ["pseudo" indicates a minus sign. Note that one could also treat spacetime as Euclidean, provided time be measured in *imaginary* units (the square-root of a negative number)]. Similarly, the famous mc^2 is the invariant pseudo-Pythagorean magnitude of energy and momentum,

$$\begin{aligned} -(mc^2)^2 &= (momentum\ in\text{"}x\text{"}direction \times velocity\ of\ light)^2 \\ &+ (momentum\ in\text{"}y\text{"}direction \times velocity\ of\ light)^2 \\ &+ (momentum\ in\text{"}z\text{"}direction \times velocity\ of\ light)^2 \\ &- (energy)^2 \end{aligned}$$

so thet mc^2 coincides with the rest energy (when all momenta vanish). Here too, the velocity of light plays the role of an exchange rate, in going from spatial to temporal units (except that here it is in the opposite direction, for reasons of dimensionality (energy is momentum × velocity).

In the beginning, Einstein was not impressed by Minkowski's geometrical interpretation. Within a short time, however, he adopted it enthusiastically. He had tried to evaluate the effect of a gravitational potential on light - say a beam from a distant star passing close to the sun. Special Relativity and Quantum Theory, when taken together, implied that the mass appearing in Newton's law of universal attraction is really the energy-content mc^2 of that mass. For a photon, it would therefore be the energy $E = h\nu$ in Planck's Quantum Theory which would enter Newton's formula. The beam would thereby be somewhat deflected and in addition, the speed of the light in that beam would accelerate during the approaching phase and decelerate when leaving the sun's neighborhood. Einstein mused - with the speed of light having just now been promoted by him to the status of a universal invariant, how could it suddenly vary so easily? The geometrical interpretation, on the other hand meant that *the presence of a gravitational field modifies the local geometry.* Einstein quizzed his friend Marcel Grossmann, who had become a geometrician. Marcel explained that changes in the coefficients of the various terms in Pythagorean formulae correspond to the effects of curvature - and were studied by Gauss and Riemann. After several years of work, partly with Grossmann, Einstein arrived in 1915 at his General Theory of Relativity [12b]. In 1905, when he produced Special Relativity. he had been in competition with Poincare, one of the two greatest mathematicians of the day. This time, it was the other one, David Hilbert, who had recently joined the race (after Einstein had "cleared the decks") - and they arrived more or less at the same time.

In the General Theory of Relativity, the physical symmetry is extended to non-inertial frames. Geometrically, *curvature* means that if we draw the frame-axes (x,y,z,ict) at two points X and X', they will not be parallel. If we want to preserve the original orientations while we move from X to X' (this is known as *parallel transport*), we shall have to use a *"covariant derivative"*, containing a *connection*. The connection is a compensating field which "remembers" the orientations of the axes in the original frame and can therefore undo the effects of curvature. The information about the gravitational field is thus coded into several functions - the *metric* (the coefficients in the Pythagorean formula, generalized to describe curved spacetime), the *frames* (describing the changing orientations at differet points) and the *connections*. In conventional General Relativity, which corresponds to a Riemannian geometry, they are all interdependent and the assignment of a metric automatically also fixes the frames and connections. In more general geometries - which may be present in the very small, when spacetime is also quantized, according to present thinking - the number of degrees of freedom of the gravitational field is larger and all three functions may be independent [13,14]

Returning to the physics, we have here the first example of a *dynamical* symmetry, i.e. one which holds throughout spacetime, even though the frames are not parallel when at different points. The existence of a dynamical symmetry requires the geometry to contain a connection field, ensuring parallel transport. In GR this is one of the roles of the gravitational field. Roughly, the equations of the theory are built according to the scheme

Geometry (curvature, metric, connection, frames) = Sources (energy, momenta, spins)

i.e. the presence of matter induces geometrical fields. One feature of a dynamical symmetry is its *universal coupling*. In General Relativity, this is the

Principle of Equivalence. Galilei had already discovered that all bodies fall with the same acceleration and ensuing velocities in the earth's gravitational field. Newton used the same concept of mass in his Second Law (Inertia) as in his Law of Universal Gravitation. In Einstein's theory this is structural: the gravitational field (the l.h.s. in the above schematic equation) couples to amd is induced by quantities defined by kinematical symmetries (the r.h.s.). Moreover, these quantities are *conserved.*.

4 Emmy Noether's Two Theorems

Quantum theory enhanced the standing of the *action* (now an operator rather than a function) since Planck's quanta are quanta of action. The action is thus what the generalized substance of the world is made of, it should be dimensionless in a "natural" system of units (since we just *count* quanta); moreover, *conjugate variables* are those variables *whose product carries the units of action* (or is dimensionless in natural units) - this is the effective content of Bohr's "Principle of Correspondence". Thus, if location and momentum are conjugate, *measuring the one affects the other* - this is the content of Heisenberg's Uncertainty Relations. Similarly, measuring the energy affects time - just as a precision measurement of time will affect the energy. Thus, if we use a symbol P to represent the measurement of momentum (which affects location), we may also think of P as representing a change of location, a displacement. Similarly, taking the letter H (for Hamilton) to represent a measurement of energy, we may also use H to represent a displacement in time (i.e. the run of History, another interpretation of the H).

Emmy Noether was a student of Felix Klein and a collaborator of Hilbert at Goettingen [10]. After the discovery of General Relativity and starting from the Action Principle, , Noether proved two key theorems *relating sym-*

metries to conservation laws.

Let us write down an "equation",

$$H \leftrightarrow Q = 0$$

Reading this "equation" from left to right, we have a *conservation law* for an observable represented by Q. Indeed, it states that H, the passage of time (=History), as applied to the measurement of the observable Q, has no effect (=0). Thus Q is unaffected by the passage of time - it is conserved. But we can also read the equation as in Hebrew, from right to left. Now remember - measuring Q modifies some variable q, conjugate to Q (i.e. Q×q = action), so we can think of the symbol Q as describing the action of *changing q*. Thus, the Hebrew-like reading of the equation says "changing q has no effect on the History of the system". This is an Invariance Principle, a symmetry. The inverse is also true: a symmetry corresponding to the system being invariant under variation of some variable q will induce the conservation of an observable Q conjugate to Q. Emmy Noether's theorem also gives a precise algorithm for the construction of Q, given q and the Action (or the Lagrangian) functions, in terms of the canonical variables of the theory.

Noether's second theorem treats the case of a dynamical symmetry. If the variable q is taken to be a function of the location, $q = q(x)$, the theory has to contain a *connection* acting locally and compensating for the loss of parallelism between relevant sets of frames and for the presence of curvature. The connection is coupled to the relevant conserved current. Physics-wise, this is an interaction with that matter current and charge: the presence of such a charge induces a field propagating from it, the curvature.

Special Relativity, taken as an application of Noether's first theorem, links the invariance of the action under the Poincare group to the conservation of energy-momentum and generalized angular momentum. General Relativity is then an example of a connection field (the gravitational field) coupled to that energy-momentum's conserved charge-current density.

5 Unitary Symmetry and the Standard Model

Symmetry is an intercultural concept and the aesthetic aspects of Einstein's theory have captivated three generations of physicists. The inherent *beauty* of this geometrical theory - the fact that it induces in the minds of the mathematically literate the same feelings that one experiences when looking at a Michaelangelo sculpture or listening to a Bach fugue - has attracted hundreds or perhaps thousands to do research in physics. I was one of those and was very disappointed, upon entering the Quantum domain, to discover that symmetry was playing a vary minor role - and especially that geometry played no role at all - in the Strong, Weak and Electromagnetic Interactions, as of 1960. Fifteen years later, the picture had changed drastically. The whole of known fundamental physics is now geometrical, in two "blocks". Gravity, on the one hand, has still not found its Quantum version and is as yet a geometrical theory (still Einstein's, highly successful "in the large", i.e. at the macroscopic level) limited to the *classical* (=nonquantum) domain. On the other hand, there is the (so called) *Standard Model*, describing all the rest (Strong, Weak and Electromagnetic, the latter two in a Unified structure). The Standard Model is also entirely geometrical and covers the entire quantum level as well as the classical (mostly relevant in electromagnetism).

The change came in as the result of experimental facts, rather than from the theoreticians' preferences. The Strong Interactions were discovered in 1932 (with the discovery of the neutron) and the Weak in 1947 (when the pion/muon confusion was cleared up). Neither seemed to have anything to do with dynamical symmetries; only Quantum Electrodynamics was shown by Weyl in 1929 to relate to such a symmetry, with the electric charge Q generated by the theory's invariance under a local change of q(x) - in this case the complex phase introduced by Quantum Mechanics. The Schroedinger amplitude $\psi(x)$ is complex, but the relevant physical quantities depend only on the probability $\psi^*\psi$, in which the phase has cancelled out. That phase is an angle in the Argand diagram for complex numbers, and changing it involves rotations SO(2) in the plane, a group whose universal covering is U(1), i.e. *unitary transformations in one complex dimension*, an "Abelian" group [rotating by an angle α first and then by β has the same result as doing it by β first and then α; this is an Abelian group. Rotations in 3 dimensions are non-abelian, i.e. the result depends on the order.].

Weyl's construction led C.N. Yang and R.L. Mills in 1953 to construct a generalization to any nonabelian Lie group, as an exercise. The model was an elegant application of the ideas in Noether's second theorem and looked like a pilot model for General Relativity, which is why R.P. Feynman started in 1956 working on its quantization - a program which was successfully accomplished by 1971 by 't Hooft and others.

In 1957, two independent sets of *experiments showed that both the Strong and the Weak Interactions involved conserved currents, i.e. symmetries.* Moreover, first cosmic rays and then particle accelerators ushered in a flood of new particle species, mostly *hadrons*, i.e. particles (suh as the proton or pion) which experience the Strong Interactions - as against *leptons* (such as the electron or neutrino) which do not. The number of different hadrons reached the hundreds. It fell to me [15], early in 1961, to discover the *order* in this mess - identifying the regularities a la Kepler or as was done for the chemical elements by Mendeleev with his Periodic Chart. Here, it was the nonabelian group SU(3) (nicknamed "The Eightfold Way") which provided a classification of the hadrons. I was also able to predict the existence of as yet undiscovered particles which would fill some empty boxes in the classification. One such case[16], which carried the day for SU(3), was the Omega-minus, discovered early in 1964 (a particle with 3 units of "strangeness" - one of the eight "charges" of SU(3), relevant to the Weak decay modes).

The next stage was structural. Why SU(3)? It took some fifty years from Mendeleev's table to its explanation by atomic structure (Rutherford, 1911). Early in 1962, I raised the possibility [17] that SU(3) might result from compositeness, with 3 basic "bricks" as constituents (now known as *quarks* u, d, s), and hadrons being made of either 3 quarks or a quark-antiquark combinations. M. Gell-Mann, who had independently found SU(3) in 1961, formalized the quark idea in 1963 [18] (also giving it its name). By 1969, experiments had confirmed the presence of (confined) quarks in protons and neutrons.

Moreover, it was found that quarks carry two sets of SU(3) charges, nick-named *flavor* and *color* (i.e. each of u, d, s can appear in 3 "colors"). Flavor is relevant to the classification and provides a good "phenomenological" dynamical model for the Strong Interactions. It also contains the weak and electric charges. Color [19] is the origin of the interquark forces, binding the quarks into hadrons - and is thus indirectly responsible for the inter-hadron forces, namely the Strong Interactions. The Unified Weak and Electromagnetic Interactions [20] involve a reducible group SU(2)×U(1), a symmetry broken "spontaneously". Together with "QCD" (Quantum Chromodynamics) the interaction induced by the color charges [19], they make up the Standard Model . The theory is a Yang-Mills geometric ("gauge") theory based on SU(3)⊗[SU(2)×U(1)] as a local symmetry, entirely in the spirit of the Erlangen program.

Theoretical speculations have raised a variety of geometrical models in which gravity would be unified with the Standard Model, in the spirit of Einstein's unsuccessful quest in his later years. One of these models - 11-dimensional ("N=8") Supergravity - appears recently to have scored a point. After having raised great hopes in 1976 and then having been discarded by 1984, it has reemerged in 1994, out of the Membrane extension of String Theory, known as "M-theory" (for "mother" theory?), a model with some additional alluring symmetry features, known as *dualities*. However, M-theory itself is still very far, both from having anything to say that might be tested experimentally and even from a derivation from some clear and coherent basic principle. Perhaps, as in many other cases, theorists will get some new clues from experiments, when the LHC accelerator at CERN will hopefully operate around 2005.

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