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INTEGRABILITY CONDITIONS, AGGREGATION OF CONSUMER DEMAND AND PROBLEMS OF GENERALIZED PROGRAMMING

by

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The theory of aggregation of economic indicators is a part of the investigations on the mathematical modeling of economic systems. The structure of the model should correspond to the whole state of economy. In ideal the relationships of macro economic model should be derived by aggregation of micro descriptions. The violathion of aggregation conditions should correspond to the switch from one model to another. We consider in this report the aggregation conditions which arise in the models of production and consumption.

1. Construction of Economic Indices

In processing statistical information, economical indices should be calculated for separable groups of goods. Let us consider a group of m products. Denote by $\mathbf{X} = (X_1, X_2, \dots, X_m)$ an arbitrary set of the products, and denote by $\mathbf{P} = (p_1, p_2, \dots, p_m)$ the vector of corresponding prices. In aggregating, we calculate economical indices $F(\mathbf{X})$ (consumption index, or utility function) and $q(\mathbf{P})$ (price index). We assume that both indices are continuous, positively uniform of the first power, and the consumption index is concave and monotonously nondecreasing with respect to each argument. For simplicity sake, we also require that the indices are smooth.

In neoclassical theory of consumption the utility function (consumption index) is constructed by means of either demand function $\mathbf{Y}(\mathbf{P}) = (Y_1(\mathbf{P}), Y_2(\mathbf{P}), \dots, Y_m(\mathbf{P}))$ or inverse demand functions $\mathbf{P}(\mathbf{X}) = (P_1(\mathbf{X}), P_2(\mathbf{X}), \dots, P_m(\mathbf{X}))$ (here $\mathbf{Y}(\mathbf{P}(\mathbf{X})) = \mathbf{X}$).

We can propose four equivalent formulations of problem connecting initial objects with economical indices.

At first, we can assume that the solution of problem on maximizing the utility function $F(\mathbf{X})$ under the budget constraint $\mathbf{P} \cdot \mathbf{X} \le \mathbf{P} \cdot \mathbf{Y}(\mathbf{P})$, $\mathbf{X} \ge 0$, is attained on the demand functions $\mathbf{Y}(\mathbf{P})$.

At second, we can connect economical indices and inverse demand functions by means of the main formula in theory of economical indices, namely

$$q(\mathbf{P}(\mathbf{X}))dF(\mathbf{X}) = \sum_{i=1}^{m} P_i(\mathbf{X})dX_i. \tag{1}$$

At third, we can use the dual formula

$$F(\mathbf{Y}(\mathbf{P}))dq(\mathbf{P}) = \sum_{i=1}^{m} Y_i(\mathbf{P})dp_i.$$

Finally, we can require that the easily treated relations

$$q(\mathbf{P})F(\mathbf{X}) \le \mathbf{P}\cdot\mathbf{X} \quad \forall \mathbf{X} \ge 0, \mathbf{P} \ge 0, \quad \text{and} \quad q(\mathbf{P})F(\mathbf{Y}(\mathbf{P})) = \mathbf{P}\cdot\mathbf{Y}(\mathbf{P}) \quad \forall \mathbf{P} \ge 0$$

hold.

If we define indices of consumption and prices in such a way, then they are mutually dual, namely

$$q(\mathbf{P}) = \inf_{\{\mathbf{X}\ge 0 | F(\mathbf{X})>0\}} \frac{\mathbf{P}\cdot\mathbf{X}}{F(\mathbf{X})}, \quad F(\mathbf{X}) = \inf_{\{\mathbf{P}\ge 0 | q(\mathbf{P})>0\}} \frac{\mathbf{P}\cdot\mathbf{X}}{q(\mathbf{P})}. \tag{2}$$

The existence of economical indices implies that flows of various goods can be regulated by financial mechanisms. Formally, in order for economical indices to exist, it is necessary and sufficient (neglecting technical peculiarities) that the inverse demand functions (or demand functions) satisfy

(a) the separability condition, namely

$$\frac{P_i(\lambda\mathbf{X})}{P_j(\lambda\mathbf{X})} = \frac{P_i(\mathbf{X})}{P_j(\mathbf{X})}$$

for any $i, j = 1, 2, \ldots$,m; $\mathbf{X} \ge 0$, and $\lambda > 0$;

(b) the Hicks law, namely

$$\sum_{i,j=1}^{m} \frac{\partial P_i(\mathbf{X})}{\partial X_j} v_i v_j < 0$$

for any vector $\mathbf{v} = (v_1, v_2, \ldots, v_m) \ne 0$ such that $\mathbf{P}(\mathbf{X}) \cdot \mathbf{v} = 0$;
and

(c) the Frobenius condition of integrability, namely

$$P_i(\mathbf{X})\left(\frac{\partial P_j(\mathbf{X})}{\partial X_k}-\frac{\partial P_k(\mathbf{X})}{\partial X_j}\right) + P_j(\mathbf{X})\left(\frac{\partial P_k(\mathbf{X})}{\partial X_i}-\frac{\partial P_i(\mathbf{X})}{\partial X_k}\right) + P_k(\mathbf{X})\left(\frac{\partial P_i(\mathbf{X})}{\partial X_j}-\frac{\partial P_j(\mathbf{X})}{\partial X_k}\right)=0$$

for any i, j, k $(1 \leq i < j < k \leq m)$ and any $\mathbf{X} > 0$. .

The separability conditions formalizes the completeness of the assortment of goods connected by the relations of substitutability and mutual complementability. The assortment of various goods is divided into groups of substitutable and mutually complementing products. The structure of the division is described by a graph. Usually a tree subgraph is selected in the graph. The tree is an important characteristic of the structure of consumer demand. It can be determined formally from the structure of either consumption index or price index.

The Hicks law can be considered as the definition of strict concavity for the differential form corresponding to the demand functions. It arose in economical literature as a specification of the "law of decreasing utility" in terms of inverse demand functions.

Unlike the Hicks law, the Frobenius condition of integrability is an equality-type condition for the demand functions (or inverse demand functions), and hence is violated under small perturbations in the norm of space C^1. The economical interpretation of this condition is a classical problem of mathematical economy.

2. Reveald Preferences Theory and Discrete Version of the Integrability Problem

Initially the violation of condition (c) on demand functions in common position seems strange for economists. They try to find equivalent formulation of this condition which has interpretation from economics theory point of view [9]. This attempts led P.Samuelson [8] to the consept of revealed preference.

Definition 1 *We say that a vector of product* $\mathbf{X}^1 \in R^m_+$ *is revealed prefered to vector* $\mathbf{X}^2 \in R^m_+$ *if and only if* $\mathbf{P}(\mathbf{X}^1)\mathbf{X}^1 \geq \mathbf{P}(\mathbf{X}^1)\mathbf{X}^2$.

P.Samuelson gives the following interpretation to this concept. The inequality $\mathbf{P}(\mathbf{X}^1)\mathbf{X}^1 \geq \mathbf{P}(\mathbf{X}^1)\mathbf{X}^2$ means that a vector of products $\mathbf{X}^2$ is not much expensive than $\mathbf{X}^1$ when prices equal to $\mathbf{P}(\mathbf{X}^1)$. So $\mathbf{X}^2$ may be chosen by consumers. But we know that consumers choose $\mathbf{X}^1$. Consequently $\mathbf{X}^1$ revealed prefered to $\mathbf{X}^2$.

H. S. Houthekker [?] found the following formulation of integrability conditions in terms of revealed preferences.

Strong axiom of revealed preferences theory. Let $\mathbf{X}^1, \mathbf{X}^2, ..., \mathbf{X}^k$ be an arbitrary set of vectors from R^m_+. If $\mathbf{P}(\mathbf{X}^1)\mathbf{X}^1 \geq \mathbf{P}(\mathbf{X}^1)\mathbf{X}^2$, $\mathbf{P}(\mathbf{X}^2)\mathbf{X}^2 \geq \mathbf{P}(\mathbf{X}^2)\mathbf{X}^3$,..., $\mathbf{P}(\mathbf{X}^{k-1})\mathbf{X}^{k-1} \geq \mathbf{P}(\mathbf{X}^{k-1})\mathbf{X}^k$, then it is impossible that $\mathbf{P}(\mathbf{X}^k)\mathbf{X}^k > \mathbf{P}(\mathbf{X}^k)\mathbf{X}^1$.

Strong axiom is convenient for the verification on consumers demand statistics. Usually we haven't information about demand functions in the whole space R^m_+. We have time series $\{\mathbf{P}^t, \mathbf{X}^t\}_{t=0}^T$, where $\mathbf{P}^t$ – a vector of prices and $\mathbf{X}^t$ – a vector of consumed products during the period t. That is we know demand functions in finite number of points.

Let's consider a discrete version of the problem about the construction of economic indices. In this case the relationship (1) trarsfer to

$$\nu_t \mathbf{P}^t \in dF(\mathbf{X}^t), \ \nu_t > 0, \ \ (t = 0, 1, ..., T). \tag{3}$$

As was shown by S. N. Afriat [?] and H. Varian [?]

we may seek index of value Without additional restrictions in the form

$$F(\mathbf{X}) = \min_{t=0,...,T} (\lambda_t \mathbf{P}^t \mathbf{X}) \tag{4}$$

where $(\lambda_0, \lambda_1, ..., \lambda_T) > 0$. It follows that the fulfilling of (3),(4) equivalent to linear system of inequalities

$$\lambda_t \mathbf{P}^t \mathbf{X}^\tau \geq \lambda_\tau \mathbf{P}^\tau \mathbf{X}^\tau, \ \ \lambda_t > 0 \ (t, \tau = 0, ..., T). \ eqno(5)$$

Linear system (5) has $(T+1)^2$ inequalities on $(T+1)$ variables $(\lambda_0, \lambda_1, ..., \lambda_T) > 0$. It was prooved that system (5) has a solution if and only if a strong axiom fulfiled on the union of rays $\{\mathbf{X} : \mu \mathbf{X}^t \mu > 0\}$. The effective algorithm is developed for the solution of system (5). This algorithm takes into account the characteristic featuers of the system (5) and has polynomial algebraic complexity of the order const $(T+1)^3$.

We note that to every solution of system (5) correspond the time series of price indices $\{\lambda_t\}_{t=0}^T$ and indices of value $\{\lambda_t \mathbf{P}^t \mathbf{X}^t\}_{t=0}^T$. We try to construct economic indices using data about food consumption in Sweden for different time periods from 1921–1938 yaers. For the existens of economic indices we must exclude from time series the data about 1933–1935 yaers. This may interpreted as changing of consumer demand stracture after the period of great economic depression. In that years the economic stracture changed and new products and requirements arisen.

3. Aggregated Description of Consumer Demand and Industry Under Violated Conditions of Integrability

Instead of one index of consumption, we search for a system of indices by using which we could construct the aggregated demand function.

Let us consider a separable group of m ultimate products. To aggregate the products $\mathbf{X} = (X_1, \ldots, X_m)$ and their prices $\mathbf{P} = (p_1, \ldots, p_m)$, we use the set of consumption indices $F_1(\mathbf{X})$, ..., $F_k(\mathbf{X})$ and price indices $Q_1(\mathbf{p})$, ..., $Q_m(\mathbf{p})$. The specific form of the functions should agree with the description of consumer demand $\mathbf{Y}(\mathbf{p})$ for the considered ultimate products. We can also describe the behaviour of consumers by means of inverse demand functions $\mathbf{P}(\mathbf{X})$.

We assume that both the demand functions and inverse demand functions satisfy the Hicks law and separability conditions. We aggregate the ultimate products under violated integrability conditions in a sufficiently small neighbourhood of the point $\mathbf{X}^0 > 0$, $\mathbf{P}(\mathbf{X}^0) > 0$. Let us specify the apriori conditions for the indices $F_j(\mathbf{X})$ and $Q_j(\mathbf{p})$.

Definition 2 . *We say that the function $F_j(\mathbf{X})$ belongs to the class $\mathbf{A}_m$ in a neighbourhood of the point $\mathbf{X}^0$ if and only if there exists a neighbourhood $U \subset R^m_+$ of $\mathbf{X}^0$ such that (1) $F_j(\mathbf{X}^0) > 0$; (2) $F_j(\mathbf{X}) \in C^2(U)$; (3) $F_j(\lambda\mathbf{X}) = \lambda F_j(\mathbf{X})$ for $\lambda > 0$, $\mathbf{X} \in U$ and $\lambda\mathbf{X} \in U$; (4) $\operatorname{grad} F_j(\mathbf{X}^0) > 0$; and (5) the inequality*

$$\sum_{i,k=1}^{m} \frac{\partial^2 F_j(\mathbf{X}^0)}{\partial X_i \partial X_k} v_i v_k < 0$$

is satisfied for any vector $\mathbf{v} = (v_1, \ldots, v_m) \neq 0$ such that $(\operatorname{grad} F_j(\mathbf{X}^0), \mathbf{v}) = 0$.

We require that the functions $F_j(\mathbf{X})$ belong to the class $\mathbf{A}_m$, and there exists an open neighbourhood V of point $\mathbf{P}(\mathbf{X}^0)$ in which the functions $Q_j(\mathbf{p})$ are positively uniform, continuously differentiable, and $Q_j(\lambda\mathbf{p}) = \lambda Q_j(\mathbf{p})$ for any $\lambda \geq 0$, $\mathbf{p} \in V$, and $\lambda\mathbf{p} \in V$.

Let us describe how the consumption indices $\mathbf{F}(\mathbf{X}) = (F_1(\mathbf{X}), \ldots, F_k(\mathbf{X}))$ and price indices $\mathbf{Q}(\mathbf{p}) = (Q_1(\mathbf{p}), \ldots, Q_k(\mathbf{p}))$ are related to the inverse demand functions. It is logical to require that the financial balance is satisfied, i.e. the cost of products $\mathbf{X}$ at the corresponding prices $\mathbf{P}(\mathbf{X})$ is equal to the cost of aggregated products $\mathbf{F}(\mathbf{X})$ at the aggregated prices $\mathbf{Q}(\mathbf{P}(\mathbf{X}))$. This implies that the inequality

$$\sum_{j=1}^{k} Q_j(\mathbf{P}(\mathbf{X})) \cdot \mathbf{F}_j(\mathbf{X}) = \mathbf{P}(\mathbf{X})\mathbf{X} \tag{6}$$

is satisfied for any $\mathbf{X} \geq 0$. Besides we require that the cost of products $\mathbf{X} \geq 0$ at arbitrary prices $\mathbf{p} \geq 0$ is not less than the cost of aggregated products $F(\mathbf{X})$ at the prices $Q(\mathbf{p})$, i.e. the inequality

$$\sum_{j=1}^{k} Q_j(\mathbf{p}) \cdot F_j(\mathbf{X}) \leq \mathbf{p} \cdot \mathbf{X} \tag{7}$$

is satisfied for any $\mathbf{X} \geq 0$ and $\mathbf{p} \geq 0$.

Under the apriori assumptions made on the indices $\mathbf{F}(\mathbf{X})$ and $\mathbf{Q}(\mathbf{p})$, according to the Kuhn–Tucker theorem, (6) and (7) are equivalent to the relations

$$\sum_{j=1}^{k} Q_j(\mathbf{P}(\mathbf{X}))\frac{\partial F_j(\mathbf{X})}{\partial X_i} = P_i(\mathbf{X}) \quad (i = 1, \ldots, m),$$

which can be written formally as

$$\sum_{j=1}^{k} Q_j(\mathbf{P}(\mathbf{X}))dF_j(\mathbf{X}) = \sum_{i=1}^{m} P_i(\mathbf{X})dX. \tag{8}$$

Relation (8) generalizes the main formula of theory of economical indices for the case $k > 1$.

Thus the aggregation of ultimate products is reduced to finding the functions $\mathbf{F}(\mathbf{X}) = (F_1(\mathbf{X}), \ldots, F_k(\mathbf{X}))$ and $\mathbf{Q}(\mathbf{P}) = (Q_1(\mathbf{P}), \ldots, Q_k(\mathbf{P}))$ which are defined in open neighbourhoods of the points $\mathbf{X}^0$ and $\mathbf{P}(\mathbf{X}^0) > 0$, respectively, satisfy the apriori conditions, and are such that relation (8) is satisfied in an open neighbourhood of the point $\mathbf{X}^0$. Since we intend to describe the situation in as agregated way as it is possible, we choose the indices $\mathbf{F}(\mathbf{X})$ and $\mathbf{Q}(\mathbf{P})$ so that to minimize their number.

Definition 3 . *Assume that a system of economical indices* $\mathbf{F}(\mathbf{X}) = (F_1(\mathbf{X}), \ldots, F_k(\mathbf{X}))$ *and* $\mathbf{Q}(\mathbf{P}) = (Q_1(\mathbf{P}), \ldots, Q_k(\mathbf{P}))$ *defined in open neighbourhoods of the points* $\mathbf{X}^0$ *and* $\mathbf{P}(\mathbf{X}^0)$*, respectively, are chosen. Aggregated inverse demand functions are the functions* $\mathbf{R}(\mathbf{F}) = (R_1(\mathbf{F}), \ldots, R_k(\mathbf{F}))$ *defined in an open neighbourhood of* $\mathbf{F}(\mathbf{X}^0)$ *and satisfying the equality* $\mathbf{R}(\mathbf{F}(\mathbf{X})) = Q(\mathbf{P}(\mathbf{X}))$ *for* $\mathbf{X}$ *belonging to an open neighbourhood of* $\mathbf{X}^0$.

Note that the existence of aggregated inverse demand functions constrains additionally the system of economical indices to be chosen. If the aggregated functions $\mathbf{R}(\mathbf{F})$ exist, (8) can be written as

$$\sum_{i=1}^{m} P_i(\mathbf{X})dX_i = \sum_{j=1}^{k} R_j(\mathbf{F}(\mathbf{X}))dF_j(\mathbf{X}). \tag{9}$$

The minimum parameter k is determined by the properties of differential form

$$\alpha = \sum_{i=1}^{m} P_i(\mathbf{X})dX_i.$$

It follows from (9) that the functions $\mathbf{F}(\mathbf{X})$ determine the variables in terms of which we can express α. The minimum number of these variables is equal to the class of α at the point $\mathbf{X}^0$ if the class is constant in a neighbourhood of $\mathbf{X}^0$. By definition, the class of α at $\mathbf{X}^0$ coincides with the rank of characteristic system for the form, namely

$$\begin{aligned} \sum_{i=1}^{m} P_i(\mathbf{X}) dX_i &= 0, \\ \sum_{j=1}^{m} \left(\frac{\partial P_i(\mathbf{X})}{\partial X_j} - \frac{\partial P_j(\mathbf{P})}{\partial X_j} \right) dX_j &= 0, \quad i = 1, 2, \ldots, m. \end{aligned}$$

Assume that the rank is constant at a neighbourhood of $\mathbf{X}^0$. Then the parameter k in (9) is not less than the class ρ of differential form α. Note that ρ can be easily calculated in terms of the algebra of external differential forms. Indeed, let us consider the sequence of external differential forms $\omega_1 = \alpha$, $\omega_2 = d\alpha$, $\omega_3 = \alpha \wedge d\alpha$, $\omega_4 = d\alpha \wedge d\alpha$, ..., $\omega_{2l} = (d\alpha)^l$, $\omega_{2l+1} = \alpha \wedge (d\alpha)^l$, ... The class ρ at the point $\mathbf{X}^0$ is equal to the least integer r such that $\omega_{r+1}(\mathbf{X}^0) = 0$.

Theorem 1 . *Assume that there exists an open neighbourhood of the point* $\mathbf{X}^0$ *in which the class of* α *is equal to* k*, and the functions* $\mathbf{P}(\mathbf{X})$ *are* n *times continuously differentiable* ($n \geq 3$) *and satisfy the Hicks law. Then there exists a collection of reduction functions* $F_1(\mathbf{X})$, ..., $F_k(\mathbf{X})$ *which belong to the class* $\mathbf{A}_m$ *in a neighbourhood of* $\mathbf{X}^0$ *and are such that (i) equality (9) is satisfied in a neighbourhood of* $\mathbf{X}^0$*, and (ii) the aggregate inverse demand functions* $\mathbf{R}(\mathbf{F})$ *are* $(n-2)$ *times continuously differentiable in a neighbourhood of the point* $\mathbf{F}(\mathbf{X}^0)$*, satisfy the Hicks law and separability conditions at the point* $\mathbf{F}(\mathbf{X}^0)$*, and* $\mathbf{R}(\mathbf{F}(\mathbf{X}^0)) > 0$.

Corollary 1 *Assume that* $\mathbf{R}(\mathbf{F})$ *are the functions constructed in the theorem. Put* $\mathbf{Q}(\mathbf{P}) = \mathbf{R}(\mathbf{F}(\mathbf{Y}(\mathbf{P})))$. *Then there exists an open neighbourhood* U_0 *of the point* $\mathbf{P}(\mathbf{X}^0)$ *in which the function* $\mathbf{Q}(\mathbf{P})$ *are positive and continuously differentiable, and* $\mathbf{Q}(\lambda\mathbf{P}) = \lambda\mathbf{Q}(\mathbf{P})$ *at* $\lambda > 0$, $\mathbf{P} \in U$, *and* $\lambda\mathbf{P} \in U$.

Thus the construction of minimum system of aggregated inverse demand functions is connected with reducing the differential form α to the least number of variables. Under the theorem conditions, the dimension of system of aggregated inverse demand functions is equal to the class of α.

We assume that the class of α is constant and equal to ρ in an open neighbourhood of the point $\mathbf{X}^0$. Denote by k_0 the minimum number of consumption indices $F_j(\mathbf{X})$ and price indices $Q_j(\mathbf{p})$, $j = 1, 2, \ldots, k_0$, which are defined in open neighbourhoods of the points $\mathbf{X}^0$ and $\mathbf{P}(\mathbf{X}^0) > 0$, respectively, and satisfy the apriori conditions and relations (8). It follows from the theorem that $k_0 \leq \rho$. On the other hand, it follows from the definition of class of α that $k_0 \geq [(\rho+1)/2]$, where $[\cdot]$ is the integer part of real number (see the Darboux theorem on the canonical form of a differential form).

Theorem 2 . *Assume that there exists an open neighbourhood of the point* $\mathbf{X}^0$ *in which the class of* α *is equal to* ρ*, and the functions* $\mathbf{P}(\mathbf{X})$ *are infinitely differentiable and satisfy the Hicks law. Then* $k_0 = [(\rho+1)/2]$.

The aggregated description of industry usually assumes that the integrability conditions are satisfied, and maximizes the product index under balance constraints. This problem is related to equilibrium market mechanisms. If the integrability conditions are violated, the demands of economical agents are not described by a single goal function. Therefore one should consider multi-criteria problems, more precisely, problems of generalized programming introduced by Yudin [6].

Denote by $E(\mathbf{l})$ the sets of vectors of ultimate products $\mathbf{X}$ which can be produced by using the vector $\mathbf{l}$ of primary resources. We assume that $E(\mathbf{l})$ is a convex set such that if $\mathbf{X} \in E(\mathbf{l})$ and $0 \leq \mathbf{Y} \leq \mathbf{X}$, then $\mathbf{Y} \in E(\mathbf{l})$. These conditions are satisfied, if, for example, $E(\mathbf{l})$ is described by the neoclassical model of inter-sector balance [2]. The demands for ultimate products are described by inverse demand functions $\mathbf{P}(\mathbf{X})$ or the differential form of demand constructed from these functions, namely

$$\alpha = \sum_{i=1}^{m} P_i(\mathbf{X}) dX_i.$$

Here we also assume that $\mathbf{P}(\mathbf{X})$ satisfy the Hicks law and separability conditions. The functions $\mathbf{P}(\mathbf{X})$ generate the field of hyperplanes $\mathbf{P}(\mathbf{X}) \cdot (\mathbf{Y} - \mathbf{X}) = 0$ defined at $\mathbf{X} \in R^m_+$. The set $E(\mathbf{l})$ and the field of hyperplanes generated by the functions $\mathbf{P}(\mathbf{X})$ define a problem of generalized programming.

Definition 4 . *A vector* $\mathbf{X}^0$ *is called the solution to the problem of generalized programming* $\{E(\mathbf{l}), \mathbf{P}(\mathbf{X})\}$ *if (1)* $\mathbf{X}^0$ *is a Pareto optimum point of* $E(\mathbf{l})$ *and (2) the hyperplane* $\mathbf{P}(\mathbf{X}^0) \cdot (\mathbf{Y} - \mathbf{X}^0) = 0$ *is supporting plane to the set* $E(\mathbf{l})$ *at the point* $\mathbf{X}^0$.

The introduced notion is quite logical. At first, it generalizes the notion of solution to a convex programming problem. Indeed, if the integrability conditions are satisfied and the consumption index $F_0(\mathbf{X})$ exists, then the solution of corresponding problem of generalized programming coincides with the solution to the convex programming problem on maximizing $F_0(\mathbf{X})$ over the set $E(\mathbf{l})$. At second, the solution to problem of generalized programming can be treated from the economical point of view. It is attained if the economical system is regulated by equilibrium market mechanisms. Indeed, it follows from the definition that $\mathbf{X}^0$

is obtained in maximizing the producer profit at the prices $\mathbf{P}(\mathbf{X}^0)$ corresponding to the structure of ultimate consumer demand $\mathbf{X}^0$.

Under the assumptions made on the set $E(1)$ and functions $\mathbf{P}(\mathbf{X})$, the solution to problem of generalized programming exists and is unique. This follows from the results by Danilov and Sotskov [1]. The convexity (strict convexity) of goal function is extended for problems of generalized programming by the weak axiom of theory of revealed preference (the Hicks law). Cutting methods and gradient methods for solving convex programming problems can be also applied for solving numerically problems of generalized programming.

Assume that the system of consumption indices $\mathbf{F}(\mathbf{X}) = (F_1(\mathbf{X}), \ldots, F_k(\mathbf{X}))$ and price indices $\mathbf{Q}(\mathbf{p}) = (Q_1(\mathbf{p}), \ldots, Q_k(\mathbf{p}))$ satisfy relations (6) and (7). Let us consider the set of accessible indices $\Gamma(1) = \{\mathbf{Z} \in R_+^k \mid \mathbf{Z} \leq \mathbf{F}(\mathbf{X}), \mathbf{X} \in E(1)\}$. It follows from the concavity of functions $F_j(\mathbf{X})$, $j = 1, \ldots, k$, and convexity of the set $E(1)$ that $\Gamma(1)$ is a convex set.

Proposition 1 . *Assume that $\mathbf{X}^0$ is a solution to the problem of generalized programming $\{E(1), \mathbf{P}(\mathbf{X})\}$, and let $\mathbf{Q}(\mathbf{P}(\mathbf{X}^0)) > 0$. Then $\mathbf{F}(\mathbf{X}^0)$ is a Pareto optimum point of the set $\Gamma(1)$.*

Usually, in analysing multi-criteria problems, the set of Pareto optimum functions is selected from the set of accessible indices. Under the proposition conditions, this is justified by that the projection of the equilibrium point $\mathbf{X}^0$ onto the criterion space belongs to the set of Pareto optimum points. If we consider the set of accessible indices $\hat{\Gamma}(1) = \{\mathbf{Z} \in R_+^i \mid \mathbf{Z} \leq \mathbf{G}(\mathbf{X}), \mathbf{X} \in E(1)\}$ with respect to a part of criteria $\mathbf{G}(\mathbf{X}) = \{F_{i_1}(\mathbf{X}), \ldots, F_{i_l}(\mathbf{X})\}$, then the projection $\mathbf{G}(\mathbf{X}^0)$ of the equilibrium point $\mathbf{X}^0$ may not be a Pareto optimum point of $\hat{\Gamma}(1)$. Thus in formulating multi-criteria problems, finance balances (6), (7), and (8) can be considered as a constraints on the system of indices.

Proposition 2 . *Assume that there exist aggregated inverse demand functions $\mathbf{R}(\mathbf{F})$ satisfying the Hicks law and separability conditions. Let $\mathbf{X}^0$ be a solution to the problem of generalized programming $\{E(1), \mathbf{P}(\mathbf{X})\}$, and let $\mathbf{R}(\mathbf{F}(\mathbf{X}^0)) > 0$. Then $\mathbf{F}(\mathbf{X}^0)$ is a solution to the problem of generalized programming $\{\Gamma(1), \mathbf{R}(\mathbf{F})\}$.*

Let us compare the systems of indices $\mathbf{F}(\mathbf{X})$ constructed in the two theorems. On the one hand, the number of indices $\mathbf{F}(\mathbf{X})$ in the second theorem is almost twice less than that in the first one. On the other hand, unlike the first theorem, the proportions $Q_1(\mathbf{P}(\mathbf{X})) : Q_2(\mathbf{P}(\mathbf{X})) : \ldots : Q_k(\mathbf{P}(\mathbf{X}))$ in the second theorem do not depend on the system of indices $\mathbf{F}(\mathbf{X})$. The latter is quite essential. "The equilibrium point" $\mathbf{F}(\mathbf{X}^0)$ for the system of indices in Theorem 1 can be determined by solving the problem of generalized programming $\{\Gamma(1), \mathbf{R}(\mathbf{F})\}$, while for the abridged system from Theorem 2 the direction of the vector $\mathbf{Q}(\mathbf{P}(\mathbf{X}))$ does not depend on $\mathbf{F}(\mathbf{X})$, and consequently one can ensure that the "projection" $\mathbf{F}(\mathbf{X}^0)$ of the equilibrium point $\mathbf{X}^0$ is a Pareto optimum point of $\Gamma(1)$. Thus the problem of generalized programming can be aggregated for only a complete system of product indices, in terms of which one can determine the aggregated inverse demand functions.

4. Duality for the inter–industry balance model

We consider system of m pure industries, producing final products $\mathbf{X}^0 = (X_1^0, ..., X_m^0)$. Suppose that demand functions for these final products satisfy integrability conditions and to these functions there corresponds index of value $F(\mathbf{X})$ and index of price $q(\mathbf{P})$.

We describe every industry with help of production function $G_j(\mathbf{X}^j, \mathbf{l}^j)$, which is continous, concave, monotonic function. Here $\mathbf{X}^j = (X_1^j, ..., X_{j-1}^j, X_{j+1}^j, ..., X_m^j)$

is a vector of costs of output of the rest industries by j–th industry; $\mathbf{l}^j = (l_1^j, ..., l_n^j)$ is a vector of primary resources spent by j–th industry.

We consider a problem of optimal distribution of primary resources $\mathbf{l} = (l_1, ..., l_n)$ between industries:

$$F(\mathbf{X}^0) \rightarrow max \tag{10}$$

$$G_j(\mathbf{X}^j, \mathbf{l}^j) - \sum_{i=0}^{m} X_i^j \geq 0, \quad (j = 1, ..., m) \tag{11}$$

$$\sum_{j=1}^{m} \mathbf{l}^j \leq \mathbf{l}, \tag{12}$$

$$\mathbf{X}^0 \geq 0, ..., \mathbf{X}^m \geq 0, \mathbf{l}^1 \geq 0, ..., \mathbf{l}^m \geq 0, \tag{13}$$

Under the condition of productivity it follows from standard economic interpretation of duality theory that equlibrium market mechanisms (that is market mechanisms under which demand for final products is equal a supply) are optimal mechanisms of resources distribution.

Function which correspond optimal value of functional in the problem (10)–(13) with the vector of primary resources $\mathbf{l} \geq 0$ is called aggregated production function $F^A(\mathbf{l})$.

We denote by $\mathbf{s} = (s_1, ..., s_n)$ a price vector for primary resources. Profit function for the j industry $\Pi_j(\mathbf{P}, \mathbf{s})$ is connected with production function by the Legender type transforms

$$\Pi_j(\mathbf{q}, \mathbf{s}) = \sup_{\mathbf{X} \geq 0, \mathbf{l} \geq 0} \left(q_j F_j(\mathbf{X}, \mathbf{l}) - \mathbf{qX} - \mathbf{sl} \right), \tag{14}$$

$$F_j(\mathbf{X}, \mathbf{l}) = \frac{1}{q_j} \inf_{\mathbf{q}^j \geq 0, \mathbf{s} \geq 0} \left(\Pi(\mathbf{q}, \mathbf{s}) + \mathbf{qX} + \mathbf{sl} \right). \tag{15}$$

Theorem 3 . *If* $\mathbf{l} > 0$ *then*

$$F^A(\mathbf{l}) = \min_{q(\mathbf{P})\geq 1, \mathbf{P}\geq 0, \mathbf{s}\geq 0} \left(\mathbf{sl} + \sum_{j=1}^{m} \Pi_j(\mathbf{P}, \mathbf{s}) \right). \qquad (16)$$

This theorem is the duality theorem for the problem (10)–(13). The aggregeted profit function $\Pi^A(, q_0)$, where q_0 is the level of price index, is connected with aggregated production function by the transforms (14),(15). So we have as a corollary of (16) that

$$\Pi^A(\mathbf{s}, q_0) = \min_{q(\mathbf{P})\geq 1, \mathbf{P}\geq 0} \left(\sum_{j=1}^{m} \Pi_j(\mathbf{P}, \mathbf{s}) \right). \qquad (17).$$

The optimization problem (17) is the extremal problem for the calculation of equilibrium prices $\mathbf{P}$.

Let's denote $E(\mathbf{l})$ the set of vectors $\mathbf{X^0}$ for which there are exist $(\mathbf{X^1}, ..., \mathbf{X^m}, \mathbf{l^1}, ..., \mathbf{l^m})$ satisfing (11)–(13). The support function of $E(\mathbf{l})$ is

$$W(\mathbf{P}, \mathbf{l}) = \sup_{\mathbf{X}\in E(\mathbf{l})} F(\mathbf{X}).$$

Then we have from Fenchel's duality theorem the following proposition.

Proposition 3 *If* $\mathbf{l} > 0$,

$$\bar{\mathbf{X}} = argmax\{F(\mathbf{X}) : \mathbf{X} \in E(\mathbf{l}\}$$

then

$$\frac{\mathbf{P}(\bar{\mathbf{X}})}{q(\mathbf{P}(\bar{\mathbf{X}}))} = argmin\{W(\mathbf{P}, \mathbf{l}) : \mathbf{P} \geq 0, q(\mathbf{P}) \geq 1\}.$$

We want to generalize the duality theorem for the case of violation of integrability conditions for consumer demand. For this aim we transform the previous proposition.

Proposition 4 *If* $\mathbf{l} > 0$,

$$\bar{\Pi} = min\{W(\mathbf{P}, \mathbf{l}) : \mathbf{P} \geq 0, q(\mathbf{P}) \geq 1\}.$$

and

$$\bar{\mathbf{P}} = argmin\{W(\mathbf{P}, \mathbf{l}) : \mathbf{P} \geq 0, q(\mathbf{P}) \geq 1\},$$

then

$$\frac{1}{\bar{\Pi}} = max\{q(\mathbf{P}) : W(\mathbf{P}, \mathbf{l}) \leq 1, \mathbf{P} \geq 0\},$$

and

$$\frac{\bar{\mathbf{P}}}{\bar{\Pi}} = max\{q(\mathbf{P}) : W(\mathbf{P}, \mathbf{l}) \leq 1, \mathbf{P} \geq 0\}.$$

Denote by

$$T(\mathbf{l}) = \{\mathbf{P} \geq 0 : W(\mathbf{P}, \mathbf{l})\}.$$

Due to definition 4 a vector $\bar{\mathbf{P}}$ is called the solution to the dual problem of generalized programming $\{T(\mathbf{L}), \mathbf{Y}(\mathbf{P})\}$ if (1) $\bar{\mathbf{P}}$ is a Pareto optimum point of $T(\mathbf{l})$ and (2) for all $\mathbf{P} \in T(\mathbf{l}$

$$\mathbf{Y}(\bar{\mathbf{P}})(\mathbf{P} - \bar{\mathbf{P}}) \leq 0.$$

Theorem 4 *If* $\mathbf{l} > 0$, *and*$^{-}$ *is the solution to the problem of generalized programming* $\{E(\mathbf{l}), \mathbf{P}(\mathbf{X})\}$ *then*

$$\frac{\mathbf{P}(\mathbf{X})}{W(\mathbf{P}(\mathbf{X}), \mathbf{l})}$$

is the solution to the problem of generalized programming $\{T(\mathbf{l}), \mathbf{Y}(\mathbf{P})\}$.

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