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SELF-ORGANIZATION OF TWO-DIMENSIONAL VORTICES AND STELLAR SYSTEMS

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SELF-ORGANIZATION OF TWO-DIMENSIONAL VORTICES AND STELLAR SYSTEMS

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Abstract

Understanding the organization of complex systems is a challenging problem in modern science. The Great Red Spot of Jupiter is a striking example of flow organization in a turbulent background. Robust vortices also persist in the earth atmosphere (e.g. the stratospheric polar vortex), or ocean, with implications in environmental issues. It has been proposed that similar vortices occured in the proto-planetary nebula, and contributed to the dust concentration at the origin of planetary formation.

Two processes could rapidly destroy an atmospheric vortex: turbulent mixing with the surrounding fluid, or the emission of Rossby waves (sustained by the Coriolis force), and we must understand why both processes are inhibited in robust vortices. The theory of Rossby solitons can explain the absence of wave emission (cf. Nezlin's talk). We rather address the problem of robustness against turbulent mixing. This robustness is a genuine property of two-dimensional turbulence, and has been found in a laboratory experiment performed in a rotating tank, showing the persistence of a single vortex in a highly turbulent shear.

We propose an explanation of this persistence in term of statistical mechanics. It relies on the conservation properties of the two-dimensional Euler equations, describing the inviscid flow: the vorticity of each fluid particle is conserved, but mixes with the surrounding fluid. The total energy is also conserved, and this constraint generally prevents complete mixing, explaining the persistence of a non-trivial flow structure. This robust equilibrium flow structure can be predicted by maximizing a mixing entropy.

A similar idea has been previously proposed by Lynden-Bell (1967) to explain the radial distribution of star density in galaxies, by a kind of turbulent process, the "rapid relaxation", describing the stellar system as an incompressible fluid in phase space. We discuss the analogies with two-dimensional turbulence.

In conclusion, old ideas of statistical mechanics can be extended beyond their usual field of application, to predict robust organization for two-dimensional turbulence or stellar systems. Although the system is chaotic, imprediscibility is limited to fine scale fluctuations, while the large scale structures of interest remain highly predictible.

Extended text (preliminary draft)

From water streams to large weather systems, turbulence occurs in nearly all fluid motions. In spite of such a turbulent motion, more or less organized vortices are often observed. This phenomenon is still poorly understood, but significant progress have been recently realized in the particular case of two-dimensional turbulence, i. e. when fluid motion is restricted to a plane rather than in the ordinary three-dimensional space.

A plane motion is more simple to study theoretically than a three-dimensional flow. This case occurs more frequently than might be expected, for instance when the fluid is contained in a rapidly rotating tank, or when the fluid is electrically conducting (liquid metal or plasma) and submitted to a magnetic field. Confining the fluid to a thin layer may also contribute to maintain a plane flow. The study of atmospheric or oceanic motion are indeed important applications of two-dimensional theories, when the fluid layer thickness is negligible with respect to the considered horizontal scales. Spiral galaxies are also flat systems, and can be approached by two-dimensional fluid model (cf. papers by Nezlin et al. and by Fridman et al.). The proto-planetary nebula is a similar system from the fluid mechanics point of view, and the presence of organized vortices would have an important role in the concentration of dust at the origin of planetary formation (Barge and Sommeria, 1995).

In order to simplify the study of two-dimensional turbulence, one often suppose that the flow is incompressible. In this case, the flow dynamics is entirely determined by a quantity defined in each point, called vorticity, which measures the rotation rate of a local fluid element (it is the curl of the velocity). The vorticity is attached to each fluid particle and conserved in its motion (in the absence of viscous effects, which is often a good approximation). A vortex is then characterized as a region of high vorticity. This description of an elementary vortex is however not sufficient to understand the organization of vortical systems. Indeed, one generally observe many vortices interacting in a very complex way, and stirred into filaments. However at the end of this process, the flow again organizes in a few simple persisting structures, like a big unique vortex or a vortex pair.

Various experiments and numerical simulations have displayed this organization process. Sommeria et al.(1986) have realized an experiment with a rapidly rotating water tank, with the purpose of reproducing structures analogous to the Great Red Spot of Jupiter. In the experiment, the flow is practically two-dimensional, due to the effect of the Coriolis force in a rapidly rotating fluid. Vortices are permanently produced by sources (water inlets) and sinks (water outlets) flush in the tank bottom, maintained by a closed circuit pumping. The pumped radial motion, combined with rotation, produces a fluid jet opposite to the rotation direction (in the frame of reference rotating with the tank).

For a sufficiently energetic jet, a unique vortex emerges. New vortices rotating in the same direction are permanently generated by sinks, but eventually merge with the main vortex. The sources produce vortices of opposite sign, but these are rapidly disrupted by turbulence. These properties are similar to what is observed in Jupiter's atmosphere, where convective plumes could excite vortices like the sources or sinks of the experiments.

How can we explain the apparition of a robust vortex amidst a flow which is itself turbulent. The turbulence seems here to play an important role, and this has suggested a statistical approach to understand the phenomena.

In 1941, Kolmogoroff has proposed a theory of turbulence in which the fluid kinetic energy is successively transfered, in cascade, toward smaller and smaller scales, where viscosity eventually dissipates energy. However this theoretical scheme does not apply to two-dimensional flows; one can indeed prove, from the equations of motion (the Euler equations) that the process of energy degradation in cascade is forbidden, so that the kinetic energy remains constant, if viscosity is weak. This absence of energy dissipation suggests a model inspired from equilibrium statistical mechanics to describe 2D turbulence.

The first attempt in this direction was due to Onsager (1949). His idea was to approach the vorticity field by an idealized system made of a large number of point particles where vorticity is confined. In other words, instead of a continuous vorticity distribution, only a finite number of points have a non-zero vorticity. These point vortices interact like a gaz of particles. The equilibrium state reached by this system after a sufficient time can be then studied by methods of statistical mechanics. Onsager has thus shown that, under certain conditions relative to the energy of the vortex gaz, the like sign vortices have a tendency to cluster. Onsager then obtained an explanation for the robustness of the large vorticity structures observed in nature. Specific calculations of equilibrium states has been made possible using the mean field approximation of Joyce and Montgomery (1973), quite justified in such hydrodynamic problems. Then the vortices interact only in a collective way: each vortex is not sensitive to interactions with individual vortices, but only to a local field (the stream function) induced by all the other vortices.

However vorticity is in reality a continuous field (except in the particular case of superfluid helium), and the approximation of such a field by a cloud of point vortices leads to inconsistencies after a long evolution time. The approach developped by Robert (1989) (see also Robert and Sommeria, 1991) directly deals with the continuous system, and thus allows to eliminate such inconsistencies (Similar ideas have been independently proposed by Kuz'min (1982) and Miller (1990)). To develop a statistical mechanics equilibrium, one has to characterize the macroscopic state of the flow, once equilibrium has been reached. For a classical gaz, for instance, the macroscopic state is characterized by pressure and volume. How to define it in two-dimensional turbulence?

One can give to our approach the following intuitive hint. Suppose that at the initial time, the vorticity is non-zero only in some region of the fluid, a patch where vorticity is uniform and has a given value, say a. As time goes on, this patch will deform in a complex way under the effect of turbulence. However its area will remain unchanged since the fluid is incompressible, and vorticity will not change, as each fluid particle preserves its vorticity in an incompressible 2D flow. One characterizes then the equilibrium state reached after a sufficiently long time by giving, at each point of the fluid domain, the probability that the measure of vorticity at that point gives the value a. This probability field is the macroscopic state of the fluid. Like in classical statistical mechanics, the macroscopic state which is actually observed is the one with a maximal entropy, the entropy being the number of microscopic states compatible with the given macroscopic state.

How are these microscopic states defined? The probability of measuring a vorticity a in a very small fluid area is denoted p, this means that in this element the area with vorticity a occupies a fraction p of the element. One can then calculate the number of ways of dispatching this vorticity if one imagines that the area element is partitioned in equal compartments. One just need to choose the number of way of choosing the compartments of non-zero vorticity. Integrating the result over the whole fluid, one gets a so-called mixing entropy, which measures the number of ways of mixing vorticity.

The maximal vorticity would naively correspond to a complete vorticity mixing, which would appear uniform with a sufficiently coarse mesh. In reality, the conservation of the fluid energy provides a constraint, and one has to determine the state of maximal mixing compatible with these constraints. One then notice that the uniform spreading of vorticity is generally impossible: this explains that we often observe, at equilibrium, a localized vorticity structure. This statistical theory of 2D turbulence has the advantage of being compatible with the dynamics of the system: the statistical equilibrium corresponds indeed to a steady flow which transports the fluctuations without further evolution. The practical determination of the statistical equilibrium is not straightforward, but efficient numerical algorithms have been developed, involving a relaxation towards equilibrium while keeping constant the conserved quantities (Robert and Sommeria, 1992, Whitaker and Turkington, 1994).

Sommeria et al. (1991) have tested the theory by numerically solving the evolution equations with high spatial resolution. A band with uniform vorticity is chosen as the initial state, separating two regions with two opposite velocities (Fig. 2). Such a band is unstable and rolls up in vortices, amplifying any small perturbation that is initially introduced. We notice the roll up of vorticity in complex filaments, illustrating the mixing process. The vortices successively merge, and if the fluid domain were unbounded, this process would repeat itself indefinitely, forming larger and larger structures. Here the evolution is limited by introducing a condition of periodicity (the velocity at the left end is equal to the velocity at the right edge), and a stationary vortex is eventually obtained: this is the equilibrium state. For an inviscid fluid, this equilibrium state would be a very fine mosaic made of the initial vorticity levels 0 and a. However the weak viscosity initially present will smooth these fluctuations and lead to a continuous vorticity field. One finds then that the theoretical relationship between vorticity and streamfunction (from which velocity is derived) is well satisfied (Fig.2B).

However this agreement with theory is only observed in vorticity containing regions, with intense stirring, typical of the vortex merging process. The system does not quite reach the global statistical equilibrium, predicted for mixing in the whole fluid domain. An equilibrium state restricted to an active subregion, surrounded by irrotational flow, is rather observed. Such a restricted equilibrium state can take the form of monopoles or dipoles, or tripoles (Robert and Rosier, 1997, Chavanis and Sommeria, 1997).

In many cases of interest, the flow does not freely evolve toward a final state, but is instead permanently driven. Some energy dissipation also takes place, but it is often a slow process, so that flow organization by inertial stirring takes place. This is the case for the Great Red Spot of Jupiter, or the above mentioned laboratory experiments in the rotating annulus. We therefore expect that turbulence drives the system towards statistical equilibrium. Adapting the ideas of linear non-equilibrium thermodynamics, we have therefore proposed equations of relaxation towards equilibrium (Robert and Sommeria, 1992, Robert and Rosier, 1997). Such equations provide a general frame for modelling the statistical effect of "subgrid scales" in the numerical simulation of 2D fluid systems. Some kind of turbulent diffusivity is generally used for that purpose. and drives the system towards a state of rest. Our turbulence models instead drives the system towards the non-trivial statistical equilibrium, which is more realistic for 2D turbulence. This approach could have interesting applications in climate modelling, in particular for the oceanic motion, for which the statistical modelling of the small scale, unresolved, eddies is a critical issue. Applications to a simplified, two-dimensional, ocean circulation model have been developed by Kazantsev et al. (1997). The same ideas could be extended to more realistic models, including the vertical structure

The same ideas can be also applied to the dynamics of galaxies. First, spiral galaxies contain a gaz component, in which geant vortices could be generated and emit spiral arms. The set of stars has also a kind of fluid behavior, but in the six-dimensional phase space, involving both velocity and position components. The flow is incompressible in this space (a consequence of the Liouville theorem), and conserves the phase space density, like vorticity in 2D fluid dynamics. For a large system like galaxies, the close star encounters are very rare, and the stars rather interact in a collective way. Each star feels the gravity field produced by the mass density (integral over velocities of the phase-space density) averaging the effects of individual stars. This is analogous to the generation of the stream function by the vorticity field. The Vlasov equations which formalizes these ideas have a close analogy with the 2D Euler equations of fluid mechanics.

The so called elliptical galaxies have a quasi-spherical structure and are weakly influenced by the gaz component. They are therefore relatively simple example of stellar systems to study. They reveal a universal radial density structure, which fits well with a statistical equilibrium state. The density decreases with altitude, like in a planetary atmosphere (but with a non-uniform gravity field). In a gaz, molecular collisions are required to reach such an equilibrium. This is not possible in elliptical galaxies, as collisions (in fact close encounters) are very rare. and reaching statistical equilibrium by such binary interactions would require much more than the age of the universe. To resolve this paradox, Lynden-Bell (1967) has proposed that equilibrium is rather reached by a collective process of "rapid relaxation". This is like a turbulent mixing in phase space, while the effect of collisions is like molecular diffusion. There is a close analogy between this rapid relaxation and the relaxation towards statistical equilibrium in vortex systems. Developing this analogy could lead to an understanding of the classification of the observed galaxy structures (Chavanis et al. 1996).

In conclusion, old ideas of statistical mechanics can be extended beyond their usual field of application, to predict robust organization for two-dimensional turbulence or stellar systems. Although the system is chaotic, imprediscibility is limited to fine scale fluctuations, while the large scale structures of interest remain highly predictible. We are seeking to apply these ideas for modelling astrophysical and geophysical systems, averaging the fine scale fluctuations, while preserving the robust properties, associated with conservation laws and kinetic constraints. This apprach is particularly interesting for climate modelling.

It involves fairly general principles, and we may consider applications to other systems, beyond fluid mechanics or stellar systems. A first ingredient is random motion of many particles. This random motion is however influenced by some global constraint, with a mean field expression. The particles do not interact with a few partners, but rather contribute in average to a local quantity (the stream function or gravity potential), which in turn influences the particles in a self-consistent way. This approach is quite standard in usual statistical mechanics, involving a set of particles. The aim of our work is to extend it to the case of a continuous medium, involving fluid parcels rather than individual particles. The transport with volume conservation by fluid motion then plays an essential role for a precise justification of the theory. It is not clear whether one can replace such properties in non physical systems, like encountered in economy.

Figure caption

Fig. 1: Example of self-organized vortices emerging in two-dimensional shear flows A) The Great Red Spot of Jupiter; B) In a rotating tank with water.

Fig. 2: Test of the statistical theory by direct numerical computations of a 2D shear flow.

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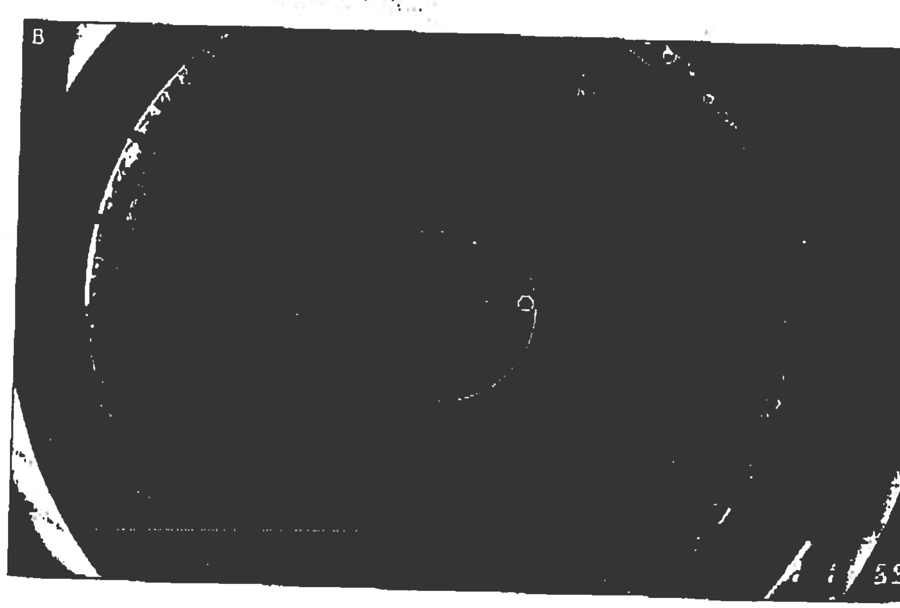
LA PERSISTANCE DES TOURBILLONS À DEUX DIMENSIONS

DES AVANCÉES THÉORIQUES PERMETTENT DE MIEUX COMPRENDRE LA TURBULENCE BIDIMENSIONNELLE. LA PERSISTANCE DE TOURBILLONS COMME LA TACHE ROUGE DE JUPITER DEVIENT MOINS MYSTÉRIEUSE.

Figure 1. Dans les écoulements turbulents, on constate souvent que des tourbillons organisés subsister malgré le désordre du fluide. Un exemple remarquable est la Tache Rouge de Jupiter, gigantesqu tourbillon atmosphérique don la taille est de l'ordre de 20 000 km et qui est présent depuis au moins trois siècles (A). Des structures persistante analogues ont été reproduite dans des expériences menées en 1988 par J. Sommeria, alc en séjour à l'université du Texas, avec une cuve en rotation rapide (3 tours par seconde) (B). Sur le cliché, pris par une caméra qui tourn avec la cuve, on voit trois couronnes d'orifices poreux pc lesquels entre et sort l'eau. Ce mouvement radial du fluide, combiné avec la rotation de la cuve, produit un jet de sens opposé à la rotation. Un tourbillon se constitue au bout d'un certain temps, qui est stable malgré la turbulence. On peut aujourd'hui rendre compte de la persistance des tourbillons dans les écoulements plans, grâce à une théorie statistique mise au point par R. Robert, à l'université de Lyon. (Cliché A : DITE/NASA ; B : auteur

Des ruisseaux aux déplacements de grandes masses d'air, la turbulence, mouvement complexe et désordonné d'un fluide, est le lot de presque tous les écoulements que l'on rencontre. Malgré la turbulence de l'écoulement, on observe couramment la présence de tourbillons plus ou moins organisés. Ce phénomène est encore mal compris, mais d'importants progrès ont été réalisés récemment dans le cas particulier de la turbulence à deux dimensions, c'est-à-dire lorsque le mouvement du fluide a lieu dans un plan et non pas dans les trois dimensions de l'espace ordinaire. Un écoulement plan a l'avantage d'être plus simple à étudier de façon théorique qu'un écoulement tridimensionnel. Le cas se rencontre plus souvent qu'on pourrait le croire ; par exemple lorsque le fluide est contenu dans un



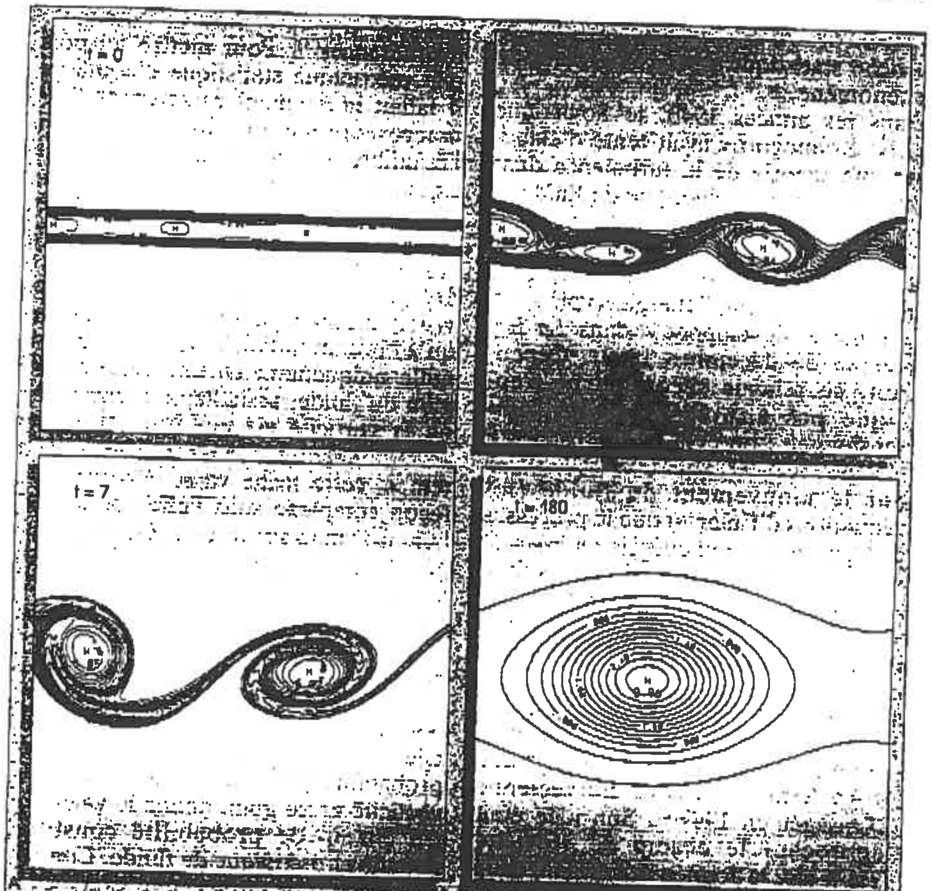
:cipient en rotation rapide, ou lorsque fluide est conducteur (métal liquide ı plasma) et soumis à un champ maıétique. De même, confiner le fluide ıns une couche de faible épaisseur :ut également contribuer à maintenir ı écoulement plan. L'étude des mou-:ents de l'atmosphère et ceux des ..ns constitue d'ailleurs une applica-ın importante des théories bidimen-ınnelles, lorsque l'épaisseur des uches fluides est négligeable par rap-rt aux échelles horizontales considé-:s.

ur simplifier l'étude de la turbulence leux dimensions, on suppose géné-ement que le fluide est incompres-le. Dans ce cas, on montre que la ıamique de l'écoulement est entière-nt déterminée par une grandeur dé-ıe en chacun des points du fluide et ɔelée vorticité, qui mesure le taux de ation propre du petit élément de de (en termes mathématiques, la ·ticité est le rotationnel de la vi-se). L'intérêt de la vorticité provient fait qu'elle est inchangée pour .que particule fluide au cours de son uvement (il faut toutefois souligner : cette propriété n'est vraie qu'en ısence de frottements visqueux, ce est souvent vérifié avec une ap-ximation raisonnable). Un tourbil-se caractérise alors localement par : vorticité élevée. Cette description n tourbillon élémentaire est cepen-.t loin de suffire pour comprendre ganisation de systèmes tourbillon-:es. En effet, on observe en général nombreux tourbillons qui inter-ssent de façon extrêmement ıplexe, et qui peuvent éclater en de tiples fragments. Mais au terme de processus, l'écoulement s'organise vent en quelques structures simples persistent — un gros tourbillon ıue, un couple de tourbillons de › opposés, etc. (voir « La danse des ·billons » dans *La Recherche* de fé-r 1990).

erses expériences et simulations nu-iques menées ces dernières années permis d'observer en détail ce pro-us d'organisation tourbillonnaire. si, par exemple, l'un d'entre nous, :ollaboration avec S.D. Meyers et .. Swinney, à l'université du Texas, alisé en 1988 une expérience avec cuve d'eau en rotation rapide, dans ıt de reproduire des structures ana-es à la Tache Rouge de Jupiter(1), : tourbillon atmosphérique observé :ette planète depuis au moins trois ; ans (fig. 1). Dans l'expérience, ıulement est pratiquement bidi-sionnel à cause de la rotation ra-de la cuve. Des tourbillons sont luits en permanence par des urces » (entrées d'eau) et des its » (sorties d'eau) disposés radialement au fond de la cuve et entretenus par une pompe en circuit fermé. Le mouvement radial d'évacuation de l'eau, combiné avec la rotation, provoque un jet de fluide qui se déplace dans le sens opposé à la rotation, si l'on se place dans le repère de la cuve.

Pour un jet suffisamment énergique, un tourbillon unique se constitue. De nouveaux tourbillons tournant dans le même sens sont créés en permanence par les puits, mais ils finissent toujours par fusionner avec le tourbillon principal. Les sources, quant à elles, produisent des tourbillons de sens contraire, mais ceux-ci sont rapidement disloqués dans la turbulence. Toutes ces propriétés ressemblent à ce qui est observé sur Jupiter, où des panaches convectifs (air chaud qui monte ou air froid qui descend) pourraient jouer un rôle analogue aux sources et aux puits de l'expérience.

Comment rendre compte de l'apparition d'un tourbillon stable au sein d'un écoulement lui-même turbulent ? La turbulence semble jouer ici un rôle de premier plan, et cela a suggéré aux



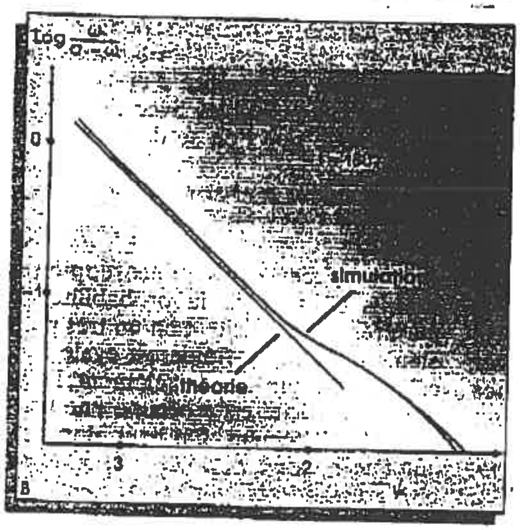


Figure 2. Pour tester leur théorie de la turbulence bidimensionnelle, les auteurs ont calculé numériquement l'évolution d'un fluide plan dont l'état initial est constitué d'une bande où la vorticité (qui mesure le taux de rotation locale du fluide) est uniforme et de valeur a, tandis que les deux régions de part et d'autre de la bande se déplacent l'une vers la droite, l'autre vers la gauche. La bande est instable et des tourbillons se forment (A). Les courbes représentent ici les lignes d'isovorticité. Un état stationnaire est atteint, là où un tourbillon unique subsiste. Dans cette structure, la relation entre la vorticité ω et la « fonction de courant » ψ (grandeur dont les dérivées spatiales donnent le champ de vitesses du fluide) vérifie très bien la loi prédite théoriquement (B).

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