Committee 1
Symmetry in Its Various Aspects:
Search for Order in the Universe

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The Two Leonardos, Part II: Leonardo da Vinci

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ABSTRACT

Part 2. Leonardo da Vinci: Symmetry in Art andNature

Picked up virtually as subliminal messages, some of the regularities and symmetries of nature have long been used by artists to organize and compose their artistic creations. A millennium before crystallography became a science, Moorish artists were decorating their architectural edifices with intuitive understanding of the space lattices. Similarly certain numbers and ratios found in nature were being incorporated -- usually unwittingly, and sometimes consciously -- by artists into their creations. Among artistic creations the*Sectio Aura* (or the "Golden Section") is found in the proportions of the pyramids of the Valley of Gizeh, in the Parthenon, in the paintings of Leonardo da Vinci, in the music of Bach and Bartock, and even in the altogether prosaic -- in three-by-five index cards. In this presentation some of the great artistic works of Western Civilization will be viewed in the light of the golden ratio.

IV. ART, ARCHITECTURE AND THE 'GOLDEN MEAN'

The Great Pyramid of Cheops

The Egyptian pyramids exhibit a number of intriguing mathematical symmetries which are worthy of mention. Their very evolution, in fact, is indicative of the builders' preoccupation with geometry. The first pyramid erected, King Zoser's 'Step Pyramid', was comprised of a set of flat *mastabas* one on top of the other and, as such, not a true pyramid. However, the next few pyramids were true pyramids, and all rising at 43^o . A pyramid of four sides and an angle of inclination of 43^o will have a perimeter to altitude ratio of 3π. Finally, the last few pyramids built, (and built no more than two centuries after Zoser's), are the pyramids in the Valley of Gizeh. These are the Cheops and the Chephron which are virtually the same size and shape. They have slopes of 52^o.

The parameters for the Cheops are well known: 230m (or 500 cubits) on each of four sides and a height of 146.4 m, (now reduced to 137m by erosion and clandestine quarry activity). For the original measurements, the base to altitude ratio is 1.57, close to the golden ratio of 1.62. Further, the ratio of the altitude of a face to one-half the length of the base, AM/OM, is exactly 1.62. A simple calculation shows that a pyramid rising at 52^o will have a base perimeter-to-altitude ratio of exactly 2π; this clearly suggests that initially a circle was laid out, and its radius was adopted for the altitude. Then the circle was "squared-off," forming the base. It is reasonable to assume that the builders' would have found this an appealing proportion, because of its simplicity. But what is far more intriguing is a computation comparing the areas of facades and base.† The base has an area of 52,900 m^2, the four sides have a combined area of 85,647 m^2. These figures can be related as follows:

$$\frac{\text{Total surface including the base}}{\text{Total surface excluding the base}} = \frac{\text{Total surface excluding the base}}{\text{Area of the base}} = 1.618,$$

which is a simple restatement of the Law of the Divine Proportion.

The unexplained question here reduces to one of 'chicken and egg.' Any pyramid in which the altitude equals the perimeter/2π, will give rise to a pyramid in which surface areas satisfy the Law of the Golden Section, and conversely, any pyramid shape which satisfies the Law of the Golden Section will have to rise at 52^o, and consequently possess a perimeter to altitude ratio of 2π. But which of these motivated the Egyptian architect's choice for the design?

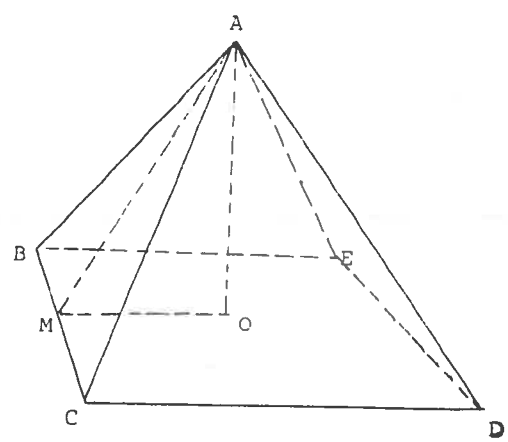

Figure 26. The original parameters of the Pyramid of Cheops had BC=CD=DE=EB=230m, AO=146 m, AM=186.2m

† This last relationship was first recognized by Johannes Kepler (1571-1630), who expressed his passion in the words: "Geometry has two great treasures; one is the theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold, the second we may name a precious jewel."

The Parthenon

Then, twenty-five centuries ago that most magnificent of all 'extrovert buildings'--the Parthenon--was also built with its facade displaying a length to width ratio of 1.618. The builders of the Parthenon undertook measures to eliminate unfavorable optical illusions by introducing a number of useful artifices. For example, a perfectly horizontal line would normally appear to sag in the middle; vertical parallel columns would appear to diverge at their tops; columns which are cylindrical would appear concave in the middle. In order to counter these effects, the Parthenon was build on a convex base of 5.7 km radius of curvature; the fluted columns, in order to avoid a splayed appearance, were sighted to converge at a point about 2.4 km high. Finally, the midsections of the columns incorporated a slight bulge, 'entasis', negating the optical illusion in the other direction. Then there is the magnificent statuary (now residing in the British Museum as the Elgin Marbles), and the majestic perch atop the Acropolis. It is the conflux of all of this along with the unassailable proportions which renders the edifice the unchallenged epitome of Classical Greek architecture--a victory for the art and science of the period.

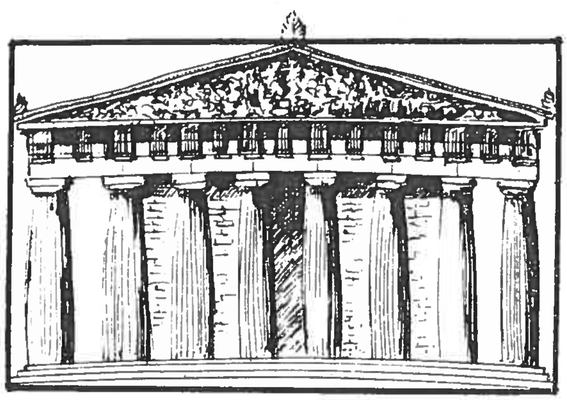

Figure 27. The Parthenon inscribed in the 'golden rectangle' and seen with the 'distortions' built in order to eliminate unpleasant optical illusions, such as sagging appearance in the middle of the base, or a splayed appearance of the columns.

In concluding this discussion on architecture, let us revisit a topic developed in Section II on polyhedra. It is possible to proceed from one type of polyhedron to another. For example, the twelve vertices of icosahedron lie on the surfaces of a cube; the eight vertices of a cube are also the vertices of a tetrahedron, etc. In the icosahedron the twelve vertices also happen to be the vertices of three golden rectangles, with their planes mutually perpendicular. This is all well and good, but so what? The preoccupation with some of these symmetries left behind a number of timeless monuments: One is the most significant of all Byzantine buildings, the Hagia Sophia in Istanbul, which displays a structure of interlocking polyhedra in the design.

Another is Kepler's well known bid to deduce the laws of planetary motion using a family of polyhedra, alternating with concentric spheres for his model (Fig. 28). The first of these is a legacy of man's supreme artistic creativity; the second, his scientific.

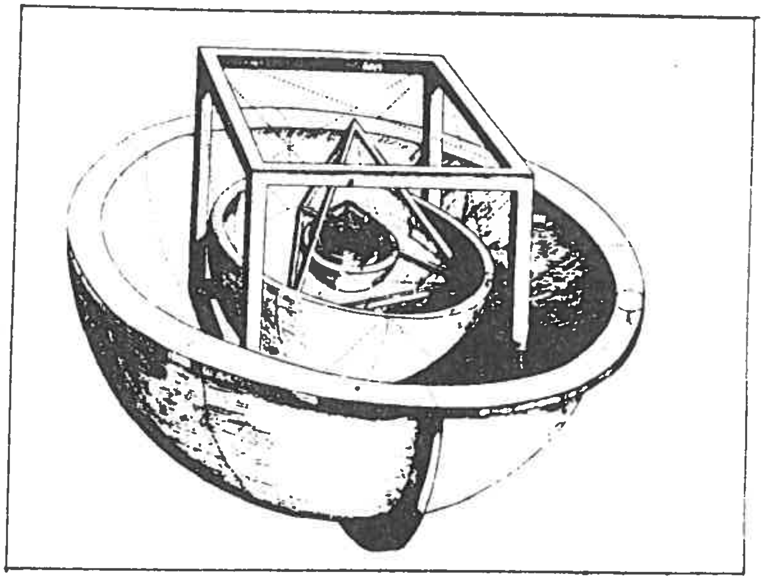

Figure 28. Kepler envisioned the orbits of planets to lie on the surfaces of six concentric spheres separated by regular polyhedra.

GRAPHIC COMPOSITION AND THE GOLDEN MEAN

The use of the golden mean has not been restricted to architecture. To the ancient Greeks it was something to be incorporated into a variety of objects, ranging from vases to eating utensils, from statuary to paintings. To the sculptors of Classical Greece and Rome, certain proportions were recognized as ideal for the human anatomy, among them the ratio of the height of person to the height of one's navel as that ubiquitorus ϕ. Again, starting from the Renaissance onward, and especially in graphic art, it has been in composition, in the construction of lines of perspective, in defining of the most salient areas of the canvas, as well as in establishing proportions that the stratagem has found use. In Figure 29, is seen the painting *the Spiridon Leda*, attributed to Leonardo da Vinci, where the ratio of the height to the height of the navel is very close to ϕ. This can be effectively dramatized by inscribing the subject of the painting in a golden rectangle, and constructing a square in the lower portion of the rectangle. The upper edge of the square is then seen to pass throught the navel.[†]

Figure 29. The Spiridon Leda by Leonardo da Vinci. The ratio of the height of the subject, *the Leda*, to the height of the navel is seen to be very close to ϕ

Figure 30. The Mona Lisa, c. 1503. The subject has been organized within an isosceles triangle of angles 36^{o} , 72^{o} and 72^{o} . The 'fleshy part,' inscribed in a golden rectangle, has the chin resting on the bottom edge of a square

Leonardo da Vinci, in his immotal painting of the Mona Lisa used an isosceles triangle with the golden proportions to organize the Mona Lisa, placing his subject's hypnotizing face near the upper vertex (Fig. 30). Moreover, when one inscribes the fleshy part of the painting in a golden rectangle and delineates a square in the upper portion of the rectangle, the chin is found to rest on the lower edge of the square, assuring that magical ratio φ. Indeed, this appears to be the technique Leonardo used in three other portraits -- those *Ginevra di Benci*, *the Lady with the Ermine,* and *La Belle Ferronier*, seen in Figures 31, 32, and 33, respectively.

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† The hypothesis of ϕ, being the ratio of one's height to the height of one's navel was put to test recently with a small group of university students, in an exercisie in taking measurements and comptuting averages and standard deviations. For the results, see Appendix A.

†† In 1509 Leonardo illustrated the book *De Divina Proportione* for his friend Luca Pacioli.

Figure 31. Portrait of Ginevra di Benci by Leonardo. A center piece of the National Gallery of Art in Washington, the painting represents the only Leonardo work to be found the United States.

Figure 32. The Lady with the Ermine by Leonardo. Krakow, Poland. As in several other portraits by the Renaissance master, the 'fleshy part, when inscribed in a golden rectangle, has the chin resting on the bottom edge of a square

Leonardo's supreme aesthetic intuition aside, just the fact that he helped to illustrate a book on the golden section by his friend Pacioli[††] suggests that in most likelihood he consciously imbued his own artwork with this formal construct also. Thus the great painting may have been a fruit of what Leonardo liked to call, 'his geometrical recreation.' In another painting, *Saint Jerome and the Lion,* (Fig. 34), Leonardo has his subject kneeling on one knee, and a position so precisely inscribable in a golden rectangle that it becomes difficult to attribute to merely an inadvertent exercise in composition.

Figure 33. *La Belle Ferroniere,* a portrait by Leonardo, hanging in the Louvre, Paris.

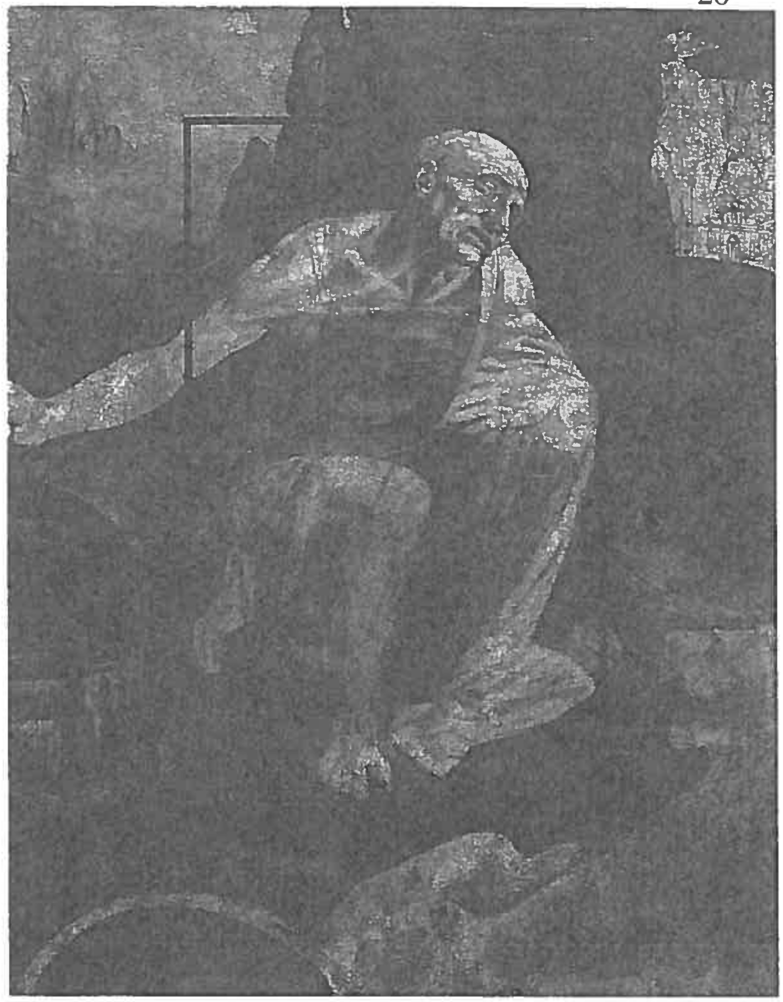

Figure 34. *St. Jerome and the Lion,* by Leonardo.

In the Middle Ages, the circle had been regarded as a symbol for 'the divine' or for heaven, (as it was for centuries by Chinese artists). The reason for this is perhaps that the circle has no beginning and no end. The earth, represented by a square with its sharply defined features, was symbolically subordinated to heaven, by having the circle circumscribe the square. The square, in turn, gave rise to isosceles triangles of angle 90 ^{o}, 45 ^{o} and 45 ^{o} . By the time of the Renaissance, the earth rose to greater prominence, but as vestiges of the older system, the triangle with a circular boundary still found some use. In the circular painting of the Holy Family by Michelangelo, and again, in the circular painting *The Madonna of the Chair by Rafael*, the compositions utilize these geometric figures. In the Michelangelo painting, a pair of interlocking triangles defines the horizon line (by way of the lower side of one triangle), and legs of the Virgin, by way of the lower side of the other triangle. The strong diagonal lines in the painting coincide with the co-linear sides of the two triangles. At the upper vertex, one finds the cluster of heads of the Infant Jesus, Joseph and Mary.

Le Corbusier, the Twentieth Century architect, felt that human life was comforted by mathematics and that the design of buildings (and even machines) should reflect the proportions of the human body. He imposed this principle on the exteriors of buildings, and on their interiors. Referring to Figures 21 and 22, the vertical and horizontal lines become virtually the full compliment of lines on the facade of one of Le Corbusier's buildings, a villa outside Paris (Figure 35). There is the vertical golden rectangle delineating the unit on the right; the other, on the lower right hand corner, representing a landing to which lead a set of stairs. Le Corbusier referred to his design technique as 'the modular system of harmonious but unequal proportions.' By any other name, it is a revival of the tradition exemplified in The Apollo Belvedere statue of two millennia past.

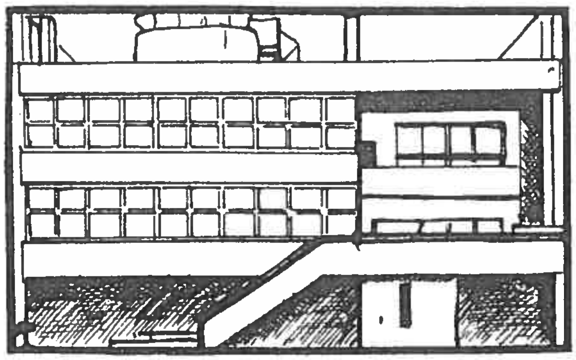

Figure 35. A sketch of le Corbusier's design for a villa in the outskirts of Paris. The facade in the sketch has been reflected right-to-left in order to show the similarities of the design with that of Figure 22.

Finally, in the two disparate schools of art, in 19th Century French Impressionism, and then in 20th Century 'Cubism' the technique of partitioning the canvas into interlocking rectangles of the golden section proportions, and delineating their associated squares is to be found. In the former school, the best proponent is George Seurat using the type of dots--*Pointilism*--in his painting called *The Parade* and again in *The Circus*; in the latter, Piet Mondrian in his work *Linear Abstractons*.

In painting a scene or a portrait, the representational artist has some freedom in raising or lowering branches, in dilating or deleting shrubbery, or even in displacing trees, but rarely the freedom in redesigning the building or the person. However, lines of perspective can be aligned with some of the diagonals within the component parts of the golden section, so that the eyes of the beholder are carried inexorably to the focal points. Inspired by the enhanced frequency with which some of the Master's have selected the focal points of their paintings, a prominent area can be identified as the one formed by the intersections of the diagonals of the golden rectangle and the diagonals of the square within the golden rectangle used in generating it (see Figures 36 and 37).

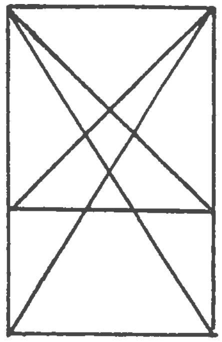

Figure 36. The Golden Rectangle and lines of perspective.

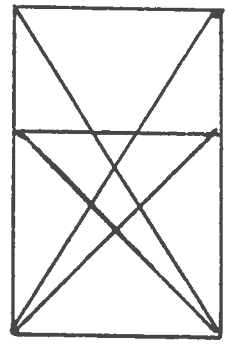

Figure 37. The Golden Rectangle and lines of perspective used in Figures 38 and 39.

The first is especially useful in the composition of portraits, the second, for street scenes. In Figures 38 and 39 are reproduced a pair of drawings (Atalay 72, 74). The first depicts 'Church Lane' in the town of Ledbury, England, and was composed utilizing the scheme of Figure 37. The second picture depicts 'The Slave Quarters' in Thomas Jefferson's Monticello. The compositions are very similar; the former,

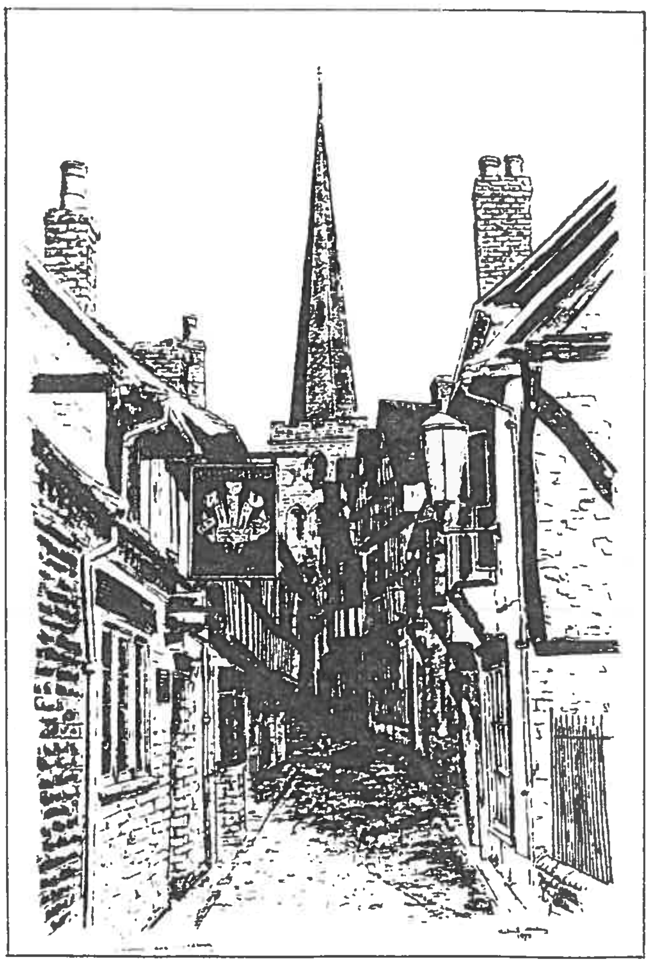

Figure 38. Church Lane, Ledbury. (Atalay 74)

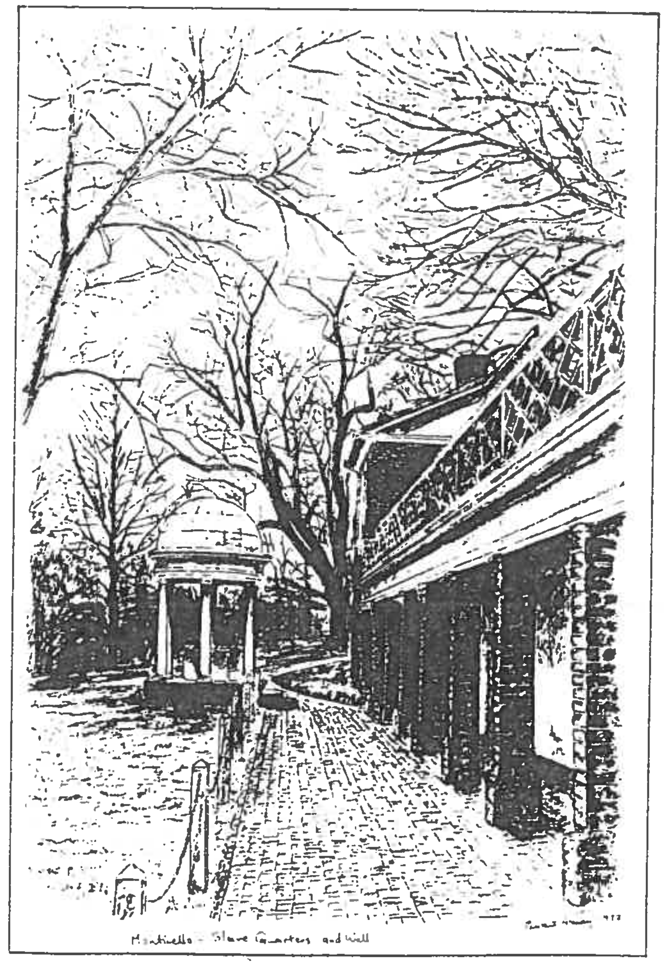

Figure 39. Slave Quarters, Monticello (Atalay 72)

although ultimately perhaps susceptible to a demonstration of equivalence. Depending on how exactly the questions are asked, the results can appear differently, but correctly. Physics abounds in examples, but Schrödinger's wave mechanics and Heisenberg's matrix mechanics, born a year apart, spring to mind immediately, their equivalence being ultimately demonstrated in the synthesis by P. A. M. Dirac.

John Wheeler's allegorical game of *Twenty Questions* (Wheeler 79) serves well to illustrate the point. In a game played in Copenhagen, all of the players who had gone before Wheeler had succeeded in guessing, in a finite amount of time, the word or subject withheld from them. When it was Wheeler's turn, not only did the other contestants seem to take an inordinate amount of time to choose a word for him, the game itself seemed to proceed at an unduly slow pace. Each time Wheeler asked a question, the other contestants would seem just as perplexed, until one of them would break the deadlock and say, 'Yes,' or 'No.' Nearing the end of his allotted twenty questions, Wheeler suddenly had a brainstorm, as he blurted out, 'It's a cloud!' Again, the others appeared confused and tentative, until, finally one of them jumped up and declared, 'Yes, he is right.' The other players came around one by one to agree that Wheeler had deduced the word correctly. When Wheeler inquired, 'What is going on? Why did you take so long in answering my questions?', they explained that for him they had not selected a word at all. Rather, they were playing along with him. The answer to his first question, whatever the question, was to be 'Yes.' After that, however, the players were on their own. Anyone who was not consistent with one of the earlier answers would have to leave the game. Thus, depending on how the questions were asked, Wheeler could have gleaned different, but correct answers. In this instance, the correct answer had been a cloud.

This may be the way scientific inquiry is carried out. One would hope that by studying physics on earth, one would pass a physics test on a planet of the star Alpha Draconis. However, the designer of the test, the taker of the test, and the grader will all have to show uncommon ingenuity and resourcefulness and understanding in seeking common bases for the 'laws' as they are deduced on each planet.

Like the artist, the physicist is a lover of nature. Just as the artist/sculptor is restricted by his imagination and his facility with his chisel or brush, the physicist is restricted only by his imagination and his facility with his mathematics. The artist is more interested in the whole of his composition than its very fine details. and the scientist is more interested in the generality of nature's laws than in its particulars. However, it is from scrutiny of a very small section of the Universe, the earth, that he tries to explain the whole. A 'beautiful law' of nature, one whose fundamental symmetries have been deciphered, one that is simple and yet general, evokes the image of an ornate tapestry, and in Feynman's words, "Nature uses only the longest threads to weave its tapestry, and each little fragment reveals the beauty of the whole thing." (Feynman 64).

The word 'Renaissance', or "Rinascimento' in Italian, (as introduced by Vasari, 65), literally means 'rebirth,' Vasari wrote of the Classical Period of Greece as representing the birth of art. Then in the Sixteenth Century, he explained, a great artistic period flowered again, culminating in a crescendo -- the High Renaissance, whose overwhelming stars were Leonardo and Michelangelo, but included luminaries such as Rafael, Titian, etc.

In rough analogy it is possible to identify the Seventeenth Century of Galileo and Newton with the birth of physics, and then the first thirty years of the Twentieth Century with its rebirth. The latter period, which was ushered in by Planck and Einstein, moreover, exhibited its own climactic High Renaissance. In the remarkably short span of about three years, between 1924 and 1927, quantum mechanics took shape in the hands of deBroglie, Schrödinger, Heisenberg, Born and Dirac. The same kind of creativity, talent and temperament which had characterized the Italian Renaissance sallied forth to produce the physics revolution. Indeed, the precise confluence of the cultural and political landscape and the availability of an extraordinary measure of talent seems to represent some of the necessary ingredients for cultural revolutions of this magnitude.

In dealing with the confluence of art and science, one last type of symmetry operation which should be mentioned is that of rescaling--of scaling-up or scaling-down. The artist, because of the physical limitations of his medium, invariably engages in enlarging or contracting of his representation. The physicist also engages in rescaling, usually in conjuring up models to represent the physical phenomenon being considered, rather than in the context of rescaling of physical structures. That physical laws are not symmetric to a change of scale is intuitively obvious to anyone who realizes that the forces which hold together a solar system, a molecule, an atom, or a nucleus are fundamentally different.

However, in the context of 'model building,' rescaling can be of considerable efficacy. In the case of the atomic nucleus, it might be useful to picture a glass ball or an infinitely hard sphere, or alternatively a gaseous ball, an oscillating liquid drop, balls connected by springs, or a football doing end-over-end rotations. Nature at this microscopic scale in reality behaves like nothing one experiences at the macroscopic level. But in different processes, one or another of these models might assist in constructing a mathematical framework to carry out calculations. Ultimately, it is the quantum mechanical description which serves in the understanding of the nucleus.

In the understanding of nature at the diametrically opposite end of the scale, at the level of stars and galaxies, useful models of space and time invariably elicit general relativity. The entire process of model building, or reducing nature to tangible, everyday pictures, however, is never irrelevant. William Blake, writing two centuries ago, unwittingly provided a timeless credo for the physicist--and so too for the artist:

To see a world in a grain of sand,
And a heaven in a wild flower.
To hold infinity in the palm of your hand,
And eternity in an hour.

From *The Auguries of Innocence*

Appendix A. Ratio of Height to Navel Height in Humans

To sculptors in Classical Greece and Rome there were ideal ratios for various body proportions. For the ratio of height-to-navel height this value was ϕ. The following data and the associated statistical computation is based on a group of twenty-one university students -- eleven females and ten males -- comprising a laboratory section of a physics class at Mary Washington College in 1990.

Height H (in cm) of 10 Males	Navel Height N (in cm)	H/N
169.0	102.0	1.657
187.5	118.5	1.582
180.0	109.0	1.643
182.0	109.0	1.669
185.0	116.0	1.594
171.0	104.0	1.644
190.5	121.0	1.574
183.0	115.0	1.574
181.0	110.5	1.634
193.5	120.0	1.612

Height H (in cm) of 11 Females	Navel Height N (in cm)	H/N
163.0	99.0	1.646
169.0	101.0	1.673
180.0	108.0	1.667
161.0	101.5	1.586
164.0	104.0	1.577
162.5	104.5	1.555
160.0	99.0	1.616
170.0	109.0	1.559
165.0	98.0	1.683
164.3	102.5	1.600
168.1	103.5	1.620

Reduction of Data

For the data above the average and standard deviation values follow:

For 10 Male students 1.620 ± 0.033

For 11 Famale students 1.616 ± 0.045

For all 21 students 1.618 ± 0.04

For reference, the values of ϕ: 1.618 034

Appendix B

The nth term of the Fibonacci Series can be computed from the closed-form expression given in the text as equation (6). The derivation of this equation follows:

Let $a_{n+2}=a_{n+1}+a_n$, so that $a_{n+2}-a_{n+1}-a_n=0$. The characteristic equation $\phi^2-\phi-1=0$, a quadratic, has characteristic solutions $\phi=(1\pm\sqrt{5})/2$.

$$a_n=c_1(\phi_1)^n+c_2(\phi_2)^n. \qquad (1')$$

$$a_0=c_1(\phi_1)^0+c_2(\phi_2)^0=1 \qquad (2')$$

$$a_1=c_1(\phi_1)^1+c_2(\phi_2)^1=1 \qquad (3')$$

From (2') we have $c_1+c_2=1$. Substituting $c_2=1-c_1$ into (3'), we obtain

$$c_1=(1+\sqrt{5})/(2\sqrt{5}) \text{ and } c_2=(\sqrt{5}-1)/(2\sqrt{5})$$

and (1') can then be written

$$a_n=(1/\sqrt{5})\{[(1+\sqrt{5})/2)]^{n+1}-[(1-\sqrt{5})/2)]^{n+1}\}$$

The terms u_1, u_2, u_3.. in the text above, are related to a_0, a_1, a_2,...in this derivation by $u_{n+1}=a_n$.

Thus

$$u_n=(1/\sqrt{5})\{[(1+\sqrt{5})/2)]^n-[(1-\sqrt{5})/2)]^n\},$$

which is just equation (6) in the text.

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