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Non-trivial Transport Phenomena: Sub- and Super-diffusion

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Non–trivial Transport Phenomena: Sub- and Super-diffusion

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1 Introduction

The main objective of this work is to inform the reader of the simple fact that there is no substuitute for the mathematical language of fractional derivatives for describing and studying the physical process of stochastic transport. Stochastic transport is now one of the most fashionable and popular fields of physics, making it possible to relate othewise very disparate phenomena, such as dissipative transport of real particles, heat, light, magnetic fields, and so on, in ordinary space, and the dynamics of point–representations of Hamiltonian machanical systems in phase space. The basis for this is the general and universal property of "forgetting" or "information loss" — the property that characterizes the stochasticity of a process.

Usual classification method for stochastic transport deals with power–law random displacements

$$l_x \propto t^\alpha \tag{1}$$

in comparison with standart diffusion law. Physisists distinguish the more rapid superdiffusion ($\alpha > 1/2$) and the slower subdiffusion ($\alpha < 1/2$) (see excellent reviews [1, 2, 3, 4]). But it is not a single difference for this topic. In theoretical research on random transport, there are two types of problem: to derive macroscopic equation for ensemble (or cloud) of particles from the stochastic microscopic law of motion of individual particles and to derive "effective" (or avereged in some sense) equation from the usual equation of convection in stochastic velocity field. All these cases give equations in fractional derivatives, but different characters.

Rather often such equation are introduced purely phenomenologically, immediately after writing down Eq. (1), and these leaves the impression that the choise of the language of fractional derivatives is arbitrary and exotic. Moreover, this method does not permit to distinguish very important and physically essential difference between mentioned types of equation. Only rigorous derivation (at least on physical level) gives an opportunity to see both an evitability and a correspondence of this "fractional" approach for real situations.

Here three problems are considered. Firstly, the derivation of macroscopic super- and subdiffusion equations in one–dimensional case. The starting point is a discrete model of classical random transport, in wich a particle executes equiprobable hops to left and right over a distance $\Delta x = 1$ in the time $\Delta t = 1$, such that at macroscopic times and scales ($t \gg 1,\ x \gg 1$) the model yields the diffusion equation for the paticle density [?]

$$\frac{\partial n}{\partial t} = \frac{1}{2}\frac{\partial^2 n}{\partial x^2}. \tag{2}$$

For real physical process, it is always possible to find appropriate quantities Δx and Δt, which differ from one phenomenon to another, but in the general mathematical approach it is convenient to employ dimensionless quantities. The physical media im which stochastic transport occurs are assumed to be uniform, isotropic, and stationary in the sense that their properties do not change in time or space (subsequent generalization are possible and not complex).

Secondly, the derivation of effective equation for transport of some substance (admimixture or "passive scalar") in two-dimensional stationary incompressible flows ($\mathbf{v}(x,y) = \{v_x, v_y\}$, $\nabla \cdot \mathbf{v} = 0$), which is described by input equation

$$\frac{\partial n}{\partial t} + \mathbf{v}\nabla n = D\Delta n, \tag{3}$$

where diffusion coefficient $D =$ const are small: $va \gg D$ (a is the characteristic spatial scale of $\mathbf{v}(\mathbf{r})$), i.e, original diffusion in (3) has a "triggering effect" [3].

It is easy to see large difference between smooth microscopic motion of the individual particles in (3) and abrupt instanteneous displacements in previous cases. Moreover, Eq. (3) quite often describes systems for which there does not exist at all any real "microscopic level" of motion (for example, scalar n may be a z-component of magnetic field [3]).

2 Macroscopic Superdiffusion

At the microscopic level, "Levy flights" provide the most faithful mathematical model of fast transport [1, 2, 3]. Here the discreteness of the hops in time is preserved, but the spatial motion becomes continuous. The spatial motion is characterized by a distribution function $f(x)$, equal to the probability distribution of a displacement at the next hop from a given point to the coordinate x. Therefore $f(x)$ is a nonnegative-definite, even (as a result of the isotropy of the medium) function, which is identical at all points in space (as a result of its uniformity), and

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1.$$

Levy flights inherently describe functions with an infinite mean-square displacement:

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 f(x)\,dx = \infty. \tag{4}$$

Thus, the outlying "tails" of f, which as a rule are assumed to be power-law functions, are responsible for superdiffusion. In what follows, to bring some degree of definiteness to the numerical coefficients, the following class of functions is used in the intermediate calculations:

$$f(x) = \frac{\Gamma(\beta + 1/2)}{\sqrt{\pi}\Gamma(\beta)} \frac{1}{(1 + x^2)^{\beta+1/2}}, \qquad \beta > 0 \tag{5}$$

where Γ is the gamma fuction. Since the final answer depends only on the power–law "tail", it will be universal and identical however different the behavior of various systems (various f) at microscopic scales $x \sim 1$. The "flights" occur for $\beta \leq 1$. One physical example of such a process is the transport of resonance radiation in a gas or plasma (Biberman–Holstein equation [5]): for a Doppler line contour $\beta \approx 1/2$, and for a Lorentzian one $\beta = 1/4$.

It is easy to see that the equation describing the dynamics of particle density for arbitrary f has the form

$$n(x,t+1) - n(x,t) = \int_{-\infty}^{+\infty} [n(x-x',t) - n(x,t)] f(x')\, dx'. \tag{6}$$

Expanding the function $n(x,t)$ (which is continuous at macroscopic times and scales) in Taylor series in x and t, and noting that $f(x)$ is even, it is easy to conclude that for finite $\langle x^2 \rangle$, Eq. (6) reduces to the diffusion equation with diffusion coefficient $D = \langle x^2 \rangle / 2$, and in the case (4) it remains an integral equation.

To study this regime in greater detail and to derive the desired mathematical formula, it is convenient to take the Fourier transform with respect to x, which transforms the convolution integral on the right–side of Eq. (6) into a product of Fourier transforms

$$\frac{\partial n_k}{\partial t} = (f_k - 1) n_k, \tag{7}$$

where for f given by Eq. (5),

$$f_k = \frac{2^{1-\beta}}{\Gamma(\beta)} k^\beta K_\beta(k)$$

and K_β is the modified Bessel function of the second kind. Since the ultimate objective is to derive a macroscopic transport equation describing the motion of a particle ensemble over large scales $x \gg 1$ ($k \ll 1$), in Eq. (7) $f_k - 1$ can be expanded in a series near $k = 0$, and only the first nonvanishing term need be retained. This yields

$$\begin{aligned} \frac{\partial n_k}{\partial t} &= -\frac{k^2}{[4(\beta-1)]} n_k, \qquad \beta > 1, \\ \frac{\partial n_k}{\partial t} &= k^2 \frac{\ln|k|}{2} n_k, \qquad \beta = 1, \\ \frac{\partial n_k}{\partial t} &= -\frac{\Gamma(1-\beta)}{\Gamma(1+\beta)} \frac{|k|^{2\beta}}{2^{2\beta}} n_k, \qquad \beta < 1. \end{aligned} \tag{8}$$

In the second case, dropping the remaining terms obviously leads to an unphysical instability for small scales ($|k| > 1$); however, this instability can easily eliminated by introducing any correction (since we are not interested in motion on small scales) that is small for $k \ll 1$ but gives the correct sign for large k. For example, $\ln|k|$ can be replaced with $\ln[|k|/(|k|+1)]$.

The last of Eq. (8), written in ordinary space as (compare Eq. (6))

$$\frac{\partial n}{\partial t} = \frac{\Gamma(\beta + 1/2)}{\sqrt{\pi}\Gamma(\beta)} P \int_{-\infty}^{+\infty} \frac{n(x')}{|x-x'|^{2\beta+1}} dx' \tag{9}$$

corresponds to superdiffusion. The expression on the right–hand side is a fractioanal derivative [6]. In multidimensional form, it is a fractional power of the Laplacian Δ^β. It is usually defined in terms of its Fourier transform (8). Naturally, manipulations with it are especially convenient in this representation, where they are technically identical to the case of a classical (local) diffusion operator. The general solution of Eq. (9)

$$n_k(t) = \exp\left(-\frac{\Gamma(1-\beta)}{\Gamma(1+\beta)}\frac{|k|^{2\beta}}{2^{2\beta}}\,t\right) n_k(0) \tag{10}$$

can be written in ordinary space in the form

$$n(x,t) = \int\limits_{-\infty}^{+\infty} G(x-x',t)n(x',0)\,dx', \tag{11}$$

where the Green's function of Eq. (9) is self–similar and is equal to

$$G(x,t) = \frac{1}{t^{1/(2\beta)}}\,\Phi\left(\frac{x}{t^{1/(2\beta)}}\right),$$

$$\Phi(\xi) = \frac{1}{\pi}\int\limits_0^\infty \exp\left(-\frac{\Gamma(1-\beta)}{\Gamma(1+\beta)}\frac{\kappa^{2\beta}}{2^{2\beta}}\right)\cos(\kappa\xi)\,d\kappa. \tag{12}$$

The Green's function itself can be found from the micriscopic description of the process [1, 2], but the macroscopic approach (11) shows much more clearly, for example, the characteristic property of stochastic transport that in the limit $t \to \infty$, when he profile $G(x)$ becomes very smooth,

$$n(x,t) \to AG(x,t), \qquad A = \int\limits_{-\infty}^{+\infty} n(x,0)\,dx.$$

This emergence into a self–similar (universal) regime is a manifestation of the property of "information loss" in a stochastic process (as compered to the case of ordinary diffusion equation [7]). It is related to the fact that, as one can see from Eq. (10), in the limit $t \to \infty$, the Green's function "cuts off" all initial harmonics with $k \neq 0$, and that annihilation of overtones is responsible for the asymptotic approach to a universal one–parameter profile. A significant difference from classical diffusion is that the corresponding self–similar profile is not Gaussian, but has instead a power–law "tail":

$$n(x,t)|_{x\to\infty} = A\,\frac{\Gamma)\beta+1/2)}{\sqrt{\pi}\Gamma(\beta)}\,\frac{t}{x^{2\beta+1}}. \tag{13}$$

(Notably, as $\beta \to 1$, its amplitude and the power of x approach finite values [1].) This can also be calculated from the inverse Fourier transform (12), i.e., once again starting from the microscopic description [1, 2], but it is simplier to see from the macroscopic equation (9). Indeed, as $x \to \infty$, the power–low kernel can be taken outside the integral, and particle conservation can be invoked:

$$\int\limits_{-\infty}^{+\infty} n(x,t)\,dx = \text{const} = A,$$

after which Eq. (13) is obtained immediately. Thus, the linear time dependence of the tail is related to the "constancy of the particle flux on large scales" — the finite probability of particles to hop from the core of the distribution immediately to the tail.

It is also interesting to note that the somewhat subtle property of the loss of positive-definiteness of Green's function (12) for unphysical values $\beta > 1$ can also be easily proved from Eq. (8) and (9), since by separating out the Laplacian

$$|k|^{2\beta} = -k^2(-|k|^{2\beta-2})$$

the function $|k|^{2\beta}$ with $\beta > 1$ can be transferred into the class of functions with $\beta < 1$, but with negative sign. Since in ordinary space the operator $\partial^2/\partial x^2$ does not change the sign of a power-law function, after repeating the macroscopic derivation of (13) we immediately find that the tail of G is negative (albeit for $\beta > 2$ we again end up in the region where its values are positive). Even these examples of the mathematical simplifications attest to the usefulness of the macroscopic "fractional" approach. It can be even more useful in specific physical problems.

In concluding this section, we point out that the convergence of n to a self-similar profile can be improved by introducing one more parametrization — a displacement $G(x - x_0)$: expanding the Green's function in Eq. (11) in a Taylor series in x and writing $\int xn(x,0)\,dx = Ax_0$. it can be shown that if the integral $\int x^2 n(x,0)\,dx$ is finite,

$$n(x,t) = AG(x - x_0)[1 + O(t^{-\beta})]. \tag{14}$$

The case of Gaussian Green's function is different, in that the next term in the expansion — the initial width $n(x,t)$ — can also be compensated by an additional displacement in time $G(t + t_0)$ [7]. Here this is not the case: the corresponding operation is possible only if the initial distribution $n(x,0)$ has symmetric power-law tail $|x|^{-2\beta-1}$. Moreover, since in this region it is nessasary to work with functions with diverging moments, there is no reason to believe that x^2 avereged over $n(x,0)$ will be finite. In general, according to the considerations indicated above, the rate of which the self-similar regime emerges is determined by behavior of $n_k(0)$ in the limit $k \to 0$: if $n_k(0) = A + iAx_0k + C|k|^\delta + \ldots$ with $\delta < 2$, then the correction term in Eq. (14) will decrease as $O(t^{-\beta\delta/2})$. For the sake of improvement of convergence, one must use other functions, not only parametrized G.

3 Macroscopic Subdiffusion

"Traps" provide the most faithful microscopic model of slow stochastic particle transport, these appears to have first been proposed in [8] (see also [1, 2, 3]). Here, the spatial hops are discrete and the temporal dynamics is continuous; specifically, there exists a distribution function $f(t)$ which is equal to the probability distribution of hops occurring to neighboring points within a time t after the initial point is reached. It is nonnegative-definite, it does not depend on x, and

$$\int_0^\infty f(t)\,dt = 1.$$

The concept of a "trap" corresponds to an infinite mean expectation transition time,

$$\langle t \rangle = \int_0^\infty t f(t)\, dt = \infty. \tag{15}$$

The power–law tail of f is therefore responsible for subdiffusion (compare the preceding section), and to make specific calculations, in what follows we choose f to br of the form

$$f(t) = \frac{1}{\gamma} \frac{1}{(1+t)^{\gamma+1}}, \qquad \gamma > 0. \tag{16}$$

Traps appear for $\gamma \leq 1$. A physical example here is charge transport in amorphous materials [1, 2], where $\gamma \approx 1/2$.

The calculations for this regime are more complicated than for superdiffusion. Since particles located at a given point in space "remember" well when they arrived at that point, here it is necessary to introduce a characteristic time τ (time of arrival at the point) and particle density distribution N at a given point over this time:

$$n(x,t) = \int_0^\infty N(x,t,\tau)\, d\tau.$$

It is also convenient to use a different notation for the probability that particles "survive" i.e., do not hop to neighboring points) to time τ:

$$F(\tau) = 1 - \int_0^\tau f(t)\, dt = \frac{1}{(1+\tau)^\gamma}.$$

In these terms

$$n(x,t) = \frac{1}{2}\int_0^t [Q(x-1,t-\tau) + Q(x+1,t-\tau)] F(\tau)\, d\tau + \int_t^\infty \frac{N_0(x,\tau-t)}{F(\tau-t)} F(\tau)\, d\tau,$$

$$Q(x,t) = \int_0^\infty \frac{N(\tau)}{F(\tau)} f(\tau)\, d\tau. \tag{17}$$

Here $Q(x,t)$ is the particle flux from a given point to neighboring points (the factor 1/2 results from the fact that the probability of a hop to the right is equal to the probability of a hop to the left), and $N_0(x,\tau)$ is the initial distribution function at the given point (initial condition). Next, for simplicity we assume that $N_0 = n_0\delta_+(\tau)$, where δ_+ is the "shifted" Dirac delta function with the normalization condition $\int_0^\infty \delta_+(\tau)\, d\tau = 1$ (for systems with "forgetting" asymptotic behaviour at $t \to \infty$ does not depend on initial details, see below). In then follows from Eq. (17) (with the using specific dependence $N(t,\tau)$) that

$$n(x,t) - \int_0^t n(x,t-\tau) f(\tau)\, d\tau =$$

$$\frac{1}{2}\int_0^t [n(x-1,t-\tau)+n(x+1,t-\tau)-2n(x,t-\tau)]f(\tau)\,d\tau + n_0(x)F(t), \tag{18}$$

which, after $n(x,t)$ is expanded in a Taylor series at macroscopic times and scales, reduces at finite $\langle\tau\rangle$ to the diffusion equation with $D = 1/(2\langle\tau\rangle)$ (the integral operator on the right–hand side is replaced by 1), and in the case of Eq. (15) it remains an integral equation.

Here it is convenient to employ an integral transform, as in the superdiffusion regime — but now the Laplace transform, not the Fourier one, and in time, not space. Then it follows from Eq. (18) that

$$\frac{pF_p}{1-pF_p}n_p = \frac{1}{2}\frac{d^2n_p}{dx^2} + \frac{F_p}{1-pF_p}n_p, \tag{19}$$

where, according to Eq. (16),

$$pF_p = p^\gamma e^p \Gamma(1-\gamma, p)$$

($\Gamma(a,b)$ is the incomplete gamma function). In the desired macroscopic description, only values $p \ll 1$ are important, and Eq. (19) reduces to

$$\begin{aligned}
\frac{p}{\gamma-1}n_p &= \frac{1}{2}\frac{d^2n_p}{dx^2} + \frac{n_0}{\gamma-1}, \qquad \gamma > 1,\\
-p\ln p\; n_p &= \frac{1}{2}\frac{d^2n_p}{dx^2} - \ln p\; n_p, \qquad \gamma = 1,\\
\Gamma(1-\gamma)p^\gamma n_p &= \frac{1}{2}\frac{d^2n_p}{dx^2} + \Gamma(1-\gamma)p^{\gamma-1}n_0, \qquad \gamma < 1.
\end{aligned} \tag{20}$$

The subdiffusion regime describes a third variant which has the following form in the physical coordinates (compare Eq. (18)

$$\frac{\partial}{\partial t}\int_0^t \frac{n(x,t')}{(t-t')^\gamma}\,dt' = \frac{1}{2}\frac{\partial^2 n}{\partial x^2} + \frac{n_0(x)}{t^\gamma}, \tag{21}$$

which is desired equation. The left–hand side contains the fractional derivative $\partial^\gamma/\partial t^\gamma$ [6], but of a different type than in the preceding section. Generally speaking, the extension of differential operators to fractional powers can be made by various methods, and the Fourier and Laplace transform languages employed here give different expressions (other variants are also known in mathematics [6]). This asymmetry of the spatial and temporal variables in physics is not surprising, since it is a manifestation of the causality principle. The rigorous derivation of the macroscopic equations which was presented in the present paper automatically takes into account this simple circumstance — in contrast to the phenomenological approach in [9], where it was proposed that the same type of fractional derivatives in x and t (of the type (21) be used to describe nondiffusion stochastic processes.

Moreover, in [9, 10], because of the qualitative nature of the arguments employed there, the last term on the right–hand side of Eq. (21), whose role is by no means merely formal — since it is responsible, for example, for the nonequivalence of systems with the same values of α discussed in the end of this section — dropped out of the corresponding equation.

It is no more difficult to perform operations with Eq. (21) that to perform similar operations with Eq. (9). After Fourier transforming with respect to x, its solutions assumes the form

$$n_{pk} = \frac{2\Gamma(1-\gamma)p^{\gamma-1}}{2\Gamma(1-\gamma)p^{\gamma}+k^2}n_{0k},$$

which in physical variables once again looks like Eq. (11) with a self–similar Green's function (compare with the derivation based on the microscopic description [1, 11])

$$G(x,t) = \frac{1}{t^{\gamma/2}}\,\Phi\left(\frac{x}{t^{\gamma/2}}\right),$$

$$\Phi(\xi) = \frac{\sqrt{2\Gamma(1-\gamma)}}{2\pi i\gamma}\int \exp\left(\zeta^{2/\gamma} - \sqrt{2\Gamma(1-\gamma)}\,|\xi|\zeta\right)d\zeta, \qquad (22)$$

where the integral in the complex ζ plane extends over the contour running from the the fourth quadrant into the first quadrant consisting of two rays at polar angles $\varphi = \pm\pi\gamma/2$. As $|\xi| \to \infty$, deforming this contour and passing it through the saddle point

$$\zeta = \left(\frac{\gamma}{2}\sqrt{2\Gamma(1-\gamma)}\,|\xi|\right)^{\gamma/(2-\gamma)},$$

we obtain (compare [11])

$$\Phi(\xi) \propto |\xi|^{(\gamma-1)/(2-\gamma)}\exp\left(-\frac{2-\gamma}{\gamma}\left[\frac{\gamma^2}{2}\Gamma(1-\gamma)\xi^2\right]^{1/(2-\gamma)}\right). \qquad (23)$$

In this variant of stochastic transport, we are once again dealing with emergence into a self–similar regime

$$n(x,t) = AG(x-x_0)[1+O(t^{-\gamma})],$$

for wich, as before, in the general case the convergence cannot be improved by shifting the origin of the time t (though, as one can see from the preceding discussion, the moments of G are finite here).

It is obvious that the proposed method of derivation macroscopic equations admits very simple and clearly understandable generalization to multidimensional cases and the presence of combined spatial and temporal "blurring" of the hops. We merely note here the curious inequivalence of physical processes (characterized simultaneously by (4) and (15)) with the same value of α (i.e., identical self–similarity) but different β and γ. For example, stochastic transport with $\alpha = \beta = \gamma = 1/2$ is not classical diffusion. For such processes the Green's function (and, therefor, the asymptotic solution) has different power–law tails $t^{\gamma}/|x|^{2\beta+1}$ in the limit $|x| \to \infty$ [9]. The most direct and clear way to see this is to repeat the derivation of (13) in the general case (the fractional derivative with respect to t of a power–law function is trivial to obtain and was calculated by Euler [6]). The apparent contradiction with the fact that the repeated application of the operator $\partial^{1/2}/\partial t^{1/2}$ to the equation

$$\frac{\partial^{1/2}n}{\partial t^{1/2}} = \Delta^{1/2}n$$

should transform this equation into Eq. (2) is removed by the presence of the term $n_0/t^{1/2}$ on the right–hand side, which prevents such a transformation — another argument in favor of rigorous derivation of the equations.

4 Turbulent Convection

The so–called avereged or effective equations probably have the greatest interest for process (3). These describe the long–time evolution of n, when the triggering diffusion, despite its smallness, can smooth the sharp gradients generated by the inhomogeneous velocity field $\mathbf{v}$ (see, for example, [3]). Naturally, the form of these equations depends on the topology of the given flow of the medium. In this paper, we analyze some special case of convective transport in some fluid when the corresponding "effective" equation is an equation in fractional derivatives. The case in point is "strip" (i.e., $\mathbf{v} = \{v(y), 0\}$) flows. This class of problems is fairly popular in the literature, first because it is often encountered in different practical situations and second because exact analytic results can be obtained (as the present paper also indicates), including answers to general questions that are important for any function $\mathbf{v}(\mathbf{r})$. the main attention is concentrated on the most rapid dynamics of the scalar n along x axis, i.e., the corresponding effective equation is one dimensional (along y axis, of course, there is ordinary diffusion; see below).

It should be mention here that the term "effective" is by no means used in the sense of a coarse or qualitative description of the transport of n but in the sense of the greater adequacy of the derived equations for practical requirements as compared with the original Eq. (3). The corresponding transition is entirely rigorous and correct.

Nevertheless, at the present time there are no examples of rigorous derivation of macroscopic equations corresponding to non–diffusion regime. On the other hand, as early as 1972 a simple example was known of superdiffusion in a strip flow in which the estimates at the microscopic level of the motion of the individual particles of the admixture are trivial [12] (it is necessary to stress once more, that in many practical situations there are no any real "particles" n, for example, for magnetic field). It is a set of contiguous flows of equal width a and constant velocity $\pm v_0$ whose sign is random, i.e., in each individual strip, the drift occurs with probability 1/2 in the positive or negative direction of the x axis independently of the sign of v in the neighboring strips.

Such a choice of $\mathbf{v}(y)$, which ensures the absence of a regular avereged drift ($\langle \mathbf{v} \rangle \equiv 0$), creating a rather good impression of "turbulence". Let us consider the motion of particles in this medium according to [12]. In the presence of diffusion motion at right angles to the system of flows (along y axis), a particle of the admixture crosses $N \sim \sqrt{Dt}/a$ of them during the time t. Because the direction of the velocity in each of the N flows is determined independently, the difference between the number of positive or negative signs of v in this sequence is $\Delta N \sim \sqrt{N}$. The corresponding unbalanced (more precisely, insufficiently balanced) drift leads, as is readily seen, to the (1) law of displacement with $\alpha = 3/4$:

$$l_x \sim v_0 t \, \frac{\Delta N}{N} \propto t^{3/4}.$$

the existence of such a clear and transparent microscopic picture offers the possibility of using the analitical advantages of the "strip" geometry mentioned above and deriving an effective transport equation. This can indeed be done; moreover, it appears that the most complicated obstacle in the way of solving the problem is the choice of the correct language for describing the "randomness" of the $\mathbf{v}$ in Eq. (3).

It is first of all necessary, as is generally accepted in this field (see [3]), to separate in (3) the density of the passive scalar into smooth and strongly oscillating (in this case with respect to y) components:

$$n = n(x) + \tilde{n}(x, y)$$

(it is follows from the final equations that $\tilde{n}/n \sim (a^2/Dt)^{1/4}$), the evolution of which is described by the equations

$$\frac{\partial \tilde{n}}{\partial t} - D\,\frac{\partial^2 \tilde{n}}{\partial y^2} = v(y)\,\frac{\partial n}{\partial x}, \tag{24}$$

$$\frac{\partial n}{\partial t} - \left\langle v(y)\,\frac{\partial \tilde{n}}{\partial x} \right\rangle = 0. \tag{25}$$

We have here omitted the second derivatives with respect to x — in (24) compared with the analogous operator with respect to y and in (25) compared with the retained superdiffusion operator. Futher, it is convenient to make a Laplace transformation with respect to the time:

$$p\tilde{n}_p - D\,\frac{\partial^2 \tilde{n}_p}{\partial y^2} = v(y)\,\frac{dn_p}{dx}, \tag{26}$$

$$pn_p - \left\langle v(y)\,\frac{\partial \tilde{n}_p}{\partial x} \right\rangle = n_0 \tag{27}$$

(it is assumed here that $\tilde{n}|_{t=0} = 0$, since asymptotically at $t \gg a^2/D$ the contribution of the initial condition to the solution (24) is nevertheless small, compare with previous section) and operate with equations precisely in this representations. By averaging over the plane in Eq. (27) we must obviously mean

$$\lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{+L} v(y)\,\frac{\partial \tilde{n}_p}{\partial x}\,dy. \tag{28}$$

The subsequent sequence of operations is very simple: the solution of (26), expressed in terms of the Green's function of this diffusion equation in the p representation,

$$\tilde{n}_p = \int_{-\infty}^{+\infty} \frac{\exp(-\sqrt{p/D}\,|y-y'|)}{2\sqrt{Dp}}\,v(y)\,dy' \cdot \frac{dn_p}{dx},$$

must be substituted in Eq. (28), and then the indicated limit in (27). In these calculations for given "random" velocity field the product $v(y')v(y)$ occurring in the double integral (over y' and y) must be understood as the correlation function of the flow velocity (and this is the correct representation for the macroscopic problem):

$$v(y')v(y) = v_0^2 f(|y-y'|),$$

which possesses the property that $\int_{-\infty}^{+\infty} f(z)\,dz$ converges rapidly at distances of order a (this is a certain generalization of the model, see [12]). Thus

$$\langle \ldots \rangle = \lim_{L\to\infty} \left(\frac{1}{2L} \int_{-L}^{+L} \int_{-\infty}^{+\infty} \frac{\exp(-\sqrt{p/D}\,|y-y'|)}{2\sqrt{Dp}}\,v_0^2 f(|y-y'|)\,dy'\,dy \cdot \frac{d^2 n_p}{dx^2} \right). \tag{29}$$

For $t \gg a^2/D$ we deduce from this (by going over from integration over y' and y to the integration over $y - y'$ and $y + y'$ — the main contribution here gives the domain $|y - y'| < a$)

$$\langle \ldots \rangle = \frac{v_0^2 a}{\sqrt{2Dp}} \frac{d^2 n_p}{dx^2},$$

and after multiplication of Eq. (27) by $\sqrt{p}$ and the inverse Laplace transformation this carries (25) into

$$\frac{\partial^2}{\partial t^2} \int\limits_0^t \frac{n(t')}{\sqrt{\pi(t-t')}} dt' - \frac{v_0^2 a}{\sqrt{2D}} \frac{\partial^2 n}{\partial x^2} = -\frac{n_0}{2\sqrt{\pi} t^{3/2}}, \tag{30}$$

i.e., into a typical equation (21), but with $\gamma = 3/2 > 1$!.

It can be seen that the representation employed for v (correlation function) differs from the representation in the microscopic problem in the beginning of this section. Very prpbably it was too strict adherence to the microscopic language that prevented the authors of [13] from solving this problem. They succeeded in deducing an effective equation, not for the process of spreading of the given cloud of admixture in a form avereged over the plane, but only for characteristic spreading od different clouds avereged over different realizations of the flows (or experiments). despite the apperent similarity of the problems, thay are in reality very far from each other. The second is usually "simpler", but, since it is not written in the usual physical space, it possesses quite different properties: in it, as a rule, we do not find the symmetry properties with respect to $\mathbf{r}$ and t that are in the original physical Eq. (3), and in [13] this is precisely the case (there the model equation is a diffusion type and is not ivariant with respect to the transformation $t \to t + \varepsilon$).

The self-similar Green's function of Eq. (30)

$$G(x,t) = \frac{2}{3\sqrt{C}\, t^{3/4}} \frac{1}{2\pi i} \int \exp(\zeta^{4/3} - |\xi|\zeta)\, d\zeta, \qquad C = \frac{v_0^2 a}{\sqrt{2D}}, \quad \xi = \frac{x}{\sqrt{C}\, t^{3/4}}$$

(compare with (22)) asymptotically for $|\xi| \to \infty$ has a form

$$G \approx \frac{1}{\sqrt{C}\, t^{3/4}} \frac{3}{4} \sqrt{2\pi}\, |\xi| \exp\left(-\frac{27}{256}\xi^4\right)$$

(see (23) too).

To conclude this section, we must consider the next important circumstance. The present method can be readily generalized to other classes of random functions $v(y)$ different from the example of Dreizin and Dykhne: if the correlation function of the velocity f has a power-law tail, ensuuring divergence of $\int_{-\infty}^{+\infty} f(z)\, dz$ at large $|z|$, then instead of Eq. (30) there arises a superdiffusion equation with different (of higher degree) fractional derivatives with respect to t. It means, that we deal here with a completely different type of superdiffusion equations with the one introduced in the second section. It appears that for general problems of the transport of a passive scalar the new type of equations is more characteristic. It is interesting, that despite the fractional nature of the time derivative (which, in general, can have any degree), to solve the initial-value problem for these equations it is necessary and sufficient to know only $n(x,y)|_{t=0}$, as in the original equation (3).

5 Conclusions

We see, that in rigorous approach to the different problems of stochastic transport, the fractional derivatives appear in natural manner and give us an opportunity of adequate description of each situation. An operation with them is little more complex then with usual diffusion equation.

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