Committee 5
Non-linear Structures in Natural Science and Economics

Draft – January 1, 2000
For Conference Distribution Only

Integrality Conditions and Foundations of Mathematical Modeling of Economy

Alexander A. Petrov
Head of Department
Computer Center of RAS
Moscow, Russia

The Twenty-second International Conference on the Unity of the Sciences
Seoul, Korea February 9-13, 2000

Integrality Conditions and Foundations of Mathematical Modeling of Economy

A. Petrov

Computing Center of RAS, Russia

Introduction

The main issue of my report is economic sense of Conditions of Integrability of Consumers Demand Function. Generally speaking these are the conditions of existence of consumption and prices indices for separated groups of goods. In formal sense Integrability Conditions are an analog of The Second Low of Thermodynamics but they are not fulfilled always. A. Shananin showed by means of numerical experiments with data on consumption statistics that Integrability Conditions were not fulfilled for period of 1932-35 when economic structures were been modified.

On other side it is shown that if the consumer demand functions satisfy the integrability conditions, then the Leontieff description of multi-product producing system regulated by equilibrium market mechanisms can be aggregated by means of the consumption index (utility function) into a production function. The latter describes the way the production index depends on the primary resources used by the producing system.

The aggregated description expresses the equilibrium between macro-parts of the economy, namely the industry proposing consumer goods and non-producing sector demanding these. The description is correct because it corresponds to detailed equilibrium between supply and demand of separate products.

Thus if the consumption index exists (and hence the dual price index also exists), then the economy is organized well, and production and consumption are agreed. The cost law reveals not only in a number of elementary exchanges, but also in macro-exchanges between the macro-parts. Regulating financial mechanisms are effective.

Therefore it is interesting to investigate economic sense of Integrability Conditions. I shell use for my task well known Neoclassical Model of Consumer Demand.

Aggregation of Income Distribution

Let us consider a group of m products. Denote by $\boldsymbol{X} = (X_1, X_2, ..., X_m)$ an arbitrary set of the products, and denote by $\boldsymbol{p} = (p_1, p_2, ..., p_m)$ the vector of corresponding prices. Assume that M social groups are selected in a society according to their stereotypes of consumer behavior. The stereotype of consumer behavior for the α-th group is described by the problem on maximizing a positive uniform function $u_\alpha(X)$ subject to the budget constraint $(p,X) \le I_\alpha$, $X \ge 0$, where I_α is the income of the α-th group used for consumption. We assume that the utility functions $u_\alpha(X)$ belong to the class $\mathbf{A}_m$ and $u_\alpha(X) = 0$ for $X \in \partial \mathbf{R}_+^m$

Denote by $y_\alpha(p)$ the normalized demand functions of the α-th group. Then its demand is $I_\alpha y_\alpha(p)$. From the point of view of the α-th group the price index is determined by the formula

$$q_\alpha(p) = \inf\left\{ \frac{(p,X)}{u_\alpha(X)} \;\middle|\; X \ge 0, u_\alpha(X) > 0 \right\}$$

Denote by I the total consumption fund of the society, namely $I = \sum_{\alpha=1}^{M} I_\alpha$

According to neoclassical theory, we assume that the way the income is distributed among the social groups depends on the prices p. Indeed, changes in price structure cause changes in social behavior of population. Real income, in particular, changes, and this results in migration from one social group to another. The distribution of income among groups also changes.

Denote by $\varphi_\alpha(p)$ the part of income of the α-th social group in the total consumption fund I, i.e. $\varphi_\alpha(p) = \frac{I_\alpha}{I}$. Assume that $\varphi_\alpha(p)$ are positively uniform functions.

Let us to calculate the total consumer demand of the society $I \cdot y(p)$ which unites the demands of social groups. It is clear that $y(p)$ satisfy the separability conditions.

<u>Proposition 1</u> The differential form of the demand can be represented as

$$y(p)dp = \sum_{\alpha=1}^{M} \frac{\varphi_\alpha(p)}{q_\alpha(p)} dq_\alpha(p) \tag{1}$$

Denote by $q(p) = (q_1(p), q_2(p), \dots, q_M(p))$ the vector of price indices calculated from the points of view of various groups. Assume that the system of functions $q(p)$ is functionally independent.

<u>Proposition 2</u>. Assume that there exist consumption index $F(X)$ and price index $Q(p)$ belonging to the class $\mathbf{A}_m$ and corresponding to the demand functions $y(p)$. Then there exists a function $\Phi(q)$ such that $Q(p) = \Phi(q(p))$ and

$$\varphi_\alpha(p) = \frac{q_\alpha(p)}{\Phi(q(p))} \frac{\partial\Phi}{\partial q_\alpha}(q(p)), \quad \alpha = 1,\dots,M \tag{2}$$

The interpretation of Proposition 1 complements that of equilibrium theory of aggregation. Now we have proved that for the integrability conditions to be satisfied for the demand functions $y(p)$, it is necessary that the distribution $\{\varphi_\alpha(p)\}$ of income among social groups depend implicitly on the prices, i.e. depend on the price indices $q(p)$ by which various social groups estimate the level of consumer prices. This implies that the distribution of income in the society should agree with the estimates of price level existing in the society. This can be interpreted in a quite logical way. Self-regulating mechanisms for distributing income should work in the society. Thus we see that mechanisms should exist for self-regulating economical processes and relations between economical agents. Economists associate them with market mechanisms. This, in particular, concerns Proposition 1. It is asserted that good markets cannot work in a normal way if labor market does not exists.

Thus assume that the distribution of income depends on the prices in terms of the indices **q**(**p**), namely $\varphi_\alpha(p) = \delta_\alpha(q(p))$, $\alpha = 1,\dots,M$.

If the distribution of income can be represented like in (2) at some function $\Phi(q)$, then the differential form of demand $(y(p), dp)$ satisfies, clearly, the integrability conditions. In order for the price index $Q(p) = \Phi(q(p))$q(p) = Φ(**q**(p)) to be continuous, convex and monotonously nondecreasing on $\mathbf{R}_+^m$ for any **q**(**p**) satisfying the same conditions, it is necessary and sufficient that the function $\Phi(q)$ also satisfies the same conditions for the economical indices.

The function $\Phi(q)$ turns out to be connected with Bergsonian Function of Welfare. Let us consider the function dual to $\Phi(q)$, namely

$$W(u) = \inf\left\{ \frac{(q,u)}{\Phi(q)} \;\middle|\; q \geq 0, \Phi(q) > 0 \right\} \tag{3}$$

where $u = (u_1, u_2, \dots, u_M)$ is the vector of utility functions of various social groups.

<u>Proposition 3</u>. Let $\Phi(q) \in \mathbf{A}_m$. Put $\hat{u}_\alpha(q) = \frac{1}{\Phi(q)} \frac{\partial\Phi}{\partial q_\alpha}(q)$, $\alpha = 1,\dots,M$. Then $\hat{u}_\alpha(q) = (\hat{u}_1(q), \hat{u}_2(q), \dots, \hat{u}_{M(q)}(q))$ is a solution to the optimization problem

$$W(u) \Rightarrow \max \quad \text{subject to} \quad (q,u) \leq 1, \quad u \geq 0.$$

In economical theory the function $W(u)$ is referred to as the Bergsonian welfare function. Let us consider how it is related to the consumption index $F(X)$.

Proposition 4. Let $\Phi(q) \in \mathbf{A}_m$, and let $\{\varphi_\alpha(p)\}$ be defined by the formula (2). Then the consumption index $F(X)$ is not less than the optimal value of the functional in the problem

$$W(u_1(X^1), u_1(X^2), \dots, u_M(X^M)) \Rightarrow \max \quad \text{subject to} \quad \sum_{\alpha=1}^{M} X^\alpha = X,$$

$$X^\alpha \geq 0, \qquad . \tag{4}$$

If $X = I \cdot y(p)$ for some $p > 0$, then $F(X)$ is equal to the optimal value of the functional in problem (4).

Usually political economy studies the problem on fair distribution of income in the society. Usually a concept of fairness is proposed to which the distribution of income should correspond. The concept is expressed formally by the Bergson function. This is a specially constructed function whose maximum is attained just at the distribution of income corresponding the concept.

The Bergson function expresses a compromise between economical interests of social groups, and can be treated as a political "party program". It seems that the program can be prescribed directly by the functions $\delta_\alpha(q(p))$, , of income distribution. But the demand functions satisfy integrability conditions corresponding functions of income distribution are generated by some Bergson function. A "party program" which do not satisfy the integrability conditions disorganizes economical system, and cannot be considered as constructive. Economical agents with rational behavior would not support such a program.

If all the social groups agree with a program, then problem (2) generates the distribution of income which ensures that the integrability conditions are satisfied for ultimate demand functions.

Finally, a social agreement generates economical structures which ensure self-organization of economical agents, allows the cost law to hold, and makes financial regulating mechanisms maximally effective.

Nonparametric Method for Analyzing Budget Statistics

To construct numerically the Bergson function, we need an initial information. Usually this is the budget statistics $\{X^{t,\alpha}, p^t \mid t = 1,...,T; \alpha = 1,...,M\}$, where p^t are prices at the time period t, and $X^{t,\alpha}$ is the consumption vector of the α-th social group at the time period t. Applying the nonparametric method for constructing economical indices (represented in the report by to the "trade statistics" $\{X^{t,\alpha}, p^t \mid t = 1,...,T; \alpha = 1,...,M\}$ of α-th social group, we can construct the price index $q_\alpha(p)$ from the point of view of the α-th social group. Put $u^t_{\alpha_t} = \frac{(p^t, X^{t,\alpha})}{q_\alpha(p)}$,. Applying the nonparametric method to $\{q(p^t), u^t \mid t = 1,...,T; u^t = (u^t_1, u^t_2,...,u^t_{M^t})\}$ considered as a trade statistics, we obtain the consumption index which is just the Bergson function generating the observed distribution of income. Thus the developed methods can be applied for analyzing budget statistics.

The obtained results explain the Bergson function in a new way. Now it not only formalize a normative concept of fair income distribution, but also characterizes real distribution of income existing in the society. If real distribution of income between social groups can be described by means of a Bergson function, then the consumption of the society as a whole can be characterized by one index, and the price level can be characterized by the price index. In this case economy is organized well, and financial mechanisms regulate effectively the distribution of resources. On the other hand, this implies that social groups achieved a compromise in distributing income.

We can also try to find a subset of social groups whose distribution of income can be described by its own Bergson function. If so, a compromise is achieved between social groups of this subset. Then the latter can be considered as a single group and characterized by one consumption index and the corresponding price index. Thus we can construct trees of social groups which characterize correctly the social structure of the society. Studying such social structures is of separate interest in making social and political decisions.