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Committee I
The Limits of Science?

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CHAOS AS A LIMITATION ON PREDICTABILITY, NOT ON SCIENCE

by

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1.Introduction

Scientific disciplines take pride on their ability to explain natural phenomena in terms of simpler concepts, and on their ability to predict the outcome of experiments that were not performed yet. In fact, the latter is an accepted test for the validity of a scientific theory. It has been therefore shocking to many thinkers to realize that a field of physics known as "chaos" combines strict scientific rigor with a claim that predictions of the temporal developments of generic natural systems are inherently impossible.

In this paper I explain the reasons for this basic restriction on the ability to predict the future. I shall exemplify the issues with very simple mathematical models. In particular I shall stress the concept of sensititivity to initial conditions, and the exponential divergence of close-by orbits. The basic notion of the impossibility of specifying initial conditions with arbitrary accuracy will be tied to the nature of real numbers.

Having done that, I shall offer some comments on why this impredictability is not, in my opinion, a limitation to science. Science, and physics in particular, are teaching us what questions about nature can be answered with precision. Since one CAN ask precise questions about chaotic systems and obtain precise answers, chaos in itself is not a limitation to science.

2. What is "chaos"?

In the vernacualr, the word "chaos" has the connotation of extreme disorder, of confusion, and of the threat of the gaping yond. This is one of the reasons that so many interested laymen were fascinated with the idea that physics can incorporate "chaos" within its rigorous domain. In physics, however, "chaos" means something very precise, having to do with the dynamical behavior of systems that are described by simple laws of motion. It means that the long time development of the dynamical system depends

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very sensitively on the preparation of its initial state. So sensitively, in fact, that there is no way to prepare the initial state of the system with such care as to allow long term predictions. That is all. The conceptual revolution has been actually in the understanding that chaos in this sense is a widespread phenomenon, and that almost all dynamical systems in nature suffer from such long term impredictability. This realization has put a very serious question mark on the accepted notion of determinism in classical mechanics. Before we comment on this issue, we should exemplify the nature of dynamical chaos with a very simple example.

Think about a process in which a variable x at every instant of time is determined by its value in the previous instant according to the equation

$$x_{n+1}= 2\, x_n \mod 1 \tag{1}$$

meaning that whatever is the value at the n'th instant, multiply this value by 2, and if the result is larger than 1, subtract 1. The initial condition x_0 must be a number between 0 and 1. For example, if $x_0= 0.372$, then $x_1=0.744$, $x_2=(1.488-1)=0.488$ etc. We refer to the sequence of values x_0, x_1, x_2,... as the "orbit" of the dynamical system (1). Evidently, this is a "deterministic" dynamical system in the sense that given an initial condition x_0 we can predict x_n for all times:

$$x_n= 2^n\, x_0 \mod 1. \tag{2}$$

Eq. (2) is correct, exact, and quite useless. This is where chaos comes in. Imagine that we start the process of iterating Eq. (1) twice, beginning with two close-by initial conditions x_0 and x_0'. Let the distance between these initial conditions be $\Delta x_0=| x_0-x_0'|$. Clearly, upon iteration, the distance between the two different orbits increases very quickly :

$$\Delta x_n=| x_n-x_n'|=2^n\Delta x_0 \tag{3}$$

Having made an arbitray small error in the specification of the initial condition x_0, very rapidly this error grows to a value of order 1, and we have no idea where on the interval (0,1) the value of x_n might be. After every iteration the error doubles in size, and even if initially it was, say, 1 part in 10,000, after 10-12 iterations it reaches the size of the unit interval.

The critical reader might shrug his shoulders and say: there is no deep problem here. Let the diligent experimenter specify his initial conditions with better precision, or maybe infinite precision, and then the problem disappears. True, and false. If the experimentrer

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COULD specify the initial conditions with infinite precision, indeed the problem of chaos as we know it in physics would disappear. However, initial conditions CANNOT be specified with infinite precision. Initial conditions are numbers. Almost all numbers are real, rather than rational. This means that in order to represent them exactly, one has to specify an infinity of digits. A typical example is the number π, which is the ratio of the circumference of the circle to the radius. Many people remember that π=3.14 and forget that this is a rather inaccurate specification. More precisely, π=3.1416, and even more precisely, π=3.141593. Exact specification of π needs an infinite number of digits, and this is of course impossible, no matter how big an apparatus is used to store or represent numbers. This is where the issue lies. Every specification of a number (or initial conditions) inolves an error, an error of truncation of an infinite amount of information, and therefore when a system suffers from sensitivity to initial conditions (i.e, it is chaotic), the long range predictabiltiy is lost.

For those familar with how numbers are represented on the computer, the issue can be further clarifed in the context of the iteration (1). On the computer numbers are represented in binary code, or to base 2. Thus every number between 0 and 1 starts with a period, and then a string of 0's and 1's, where the length of this string depends on the price paid for the computer (8 bit, 16 bit, 32 bit etc). Imaginge that we have 16 bit accuracy. Then our initial number might be .1001110110001001 . Upon multiplying by 2, the digit moves one position to the right, and if we find a 1 to the left of the period, we drop it (this is the meaning of mod 1). Unfortunately, after one iteration, we have only 15 digits in our number that arise from initial knowledge, after 2 iterations 14, and after 16 iteration we lost all knowledge of our initial conditions. Our state of ignorance about the precise initial conditions has moved from the "insignificant" digits to the "significant" ones, and we lose our ability to predict the state of the system. This is chaos in the dynamical sense.

3. Can one ask precise questions about chaotic systems?

The examples given above might leave the reader with the impression that chaotic sytems cannot be distinguished from random ones. After all, if we lose predicatibility in finite time, and from some point on we see an orbit which has nothing to do with our initial conditions, how can we distinguish the orbit from a random string of numbers? Indeed this question was one of the most improtant ones in the phase of development of our understanding of chaotic systems. It turns out that we CAN

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distinguish a chaotic from a random system. The point is, of course, that chaotic dynamics stems from a deterministic system. This fact can be recognized from an examination of the orbit , even though we cannot PREDICT the orbit. Think again about the expmple Eq (1). Suppose that we are handed a list of numbers $\{x_i\}$ i=1,2,...,n. How can we discover that it stems from the iteration of Eq (1)? Very simple! Draw a graph of X_{n+1} in terms of x_n. In other words, use coordinates such that the abscissa is x_n and the ordinate is x_{n+1}. All the points in such a plot will lie on the the graph of the function 2x mod 1 (see Fig.1a). Although we cannot predict x_n for high value of n, still x_{n+1} IS 2 x_n mod 1. We discover that the dynamical process derives from a deterministic law in one dynamical variable, and is not a set of random numbers. If the latter were the case, the 2 dimensional square in Fig 1a would be uniformly filled with points, see Fig1. b, since there is no correlation between the values of x_n and x_{n+1}. We have learned a very important lesson. Chaotic systems are dynamical systems, usually having a finite number of degrees of freedom, and therefore they can be distinguished from random systems with the help of the DIMENSION of the set on which the orbit is lying. In the present example the set is of dimension one, and in general the dimension will be higher, but finite. Random systems are infinite dimensional, in the sense that if we plot a d-dimensional diagram in which first coordinate is x_n, the second x_{n-1} , ...,the d'th x_{n-d+1} , than random signals would feel up such a volume with a cloud of points for any d . Chaotic signals will only feel a finite dimensional subvolume.

There are other questions that can be asked about chaotic systems. What is the long time average of the dynamical variable? Statistical correlations? There is a field of matematics called ergodic theory, which is suitable for giving precise answers to such questions. What we have learned about chaotic sytems is what ARE the questions that can be asked for which there is a precise answer.

4. Is chaos a limitation to science?

The answer to this question depends of course on our expectations from science. If our expectation is that science would predict the future for us, then of course chaos is a ruinous limitation to science. But this is not a reaonable expectation, and in fact not what science is all about. Science in general, and physics in particular, are not describing nature as it is out there. Physics is DEFINING what is our concept of nature, about which we can "do" physics. Nature out there is a an extremely rich, complex and overpowering entity. What physics does is to take a piece of

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this entity, isolate this piece so much as to alter its nature enough as to make it available to human study. By doing this, physics RECREATES a nature that is NOT the one out there, but a nature for which we can ask precise questions and get precise answers. The essence of physics is to discover what are such questions, and what are the sections of nature that can be so controlled. As an example, think of classical mechanics. We describe the dynamics of particles by writing differential equations and solving them on the computer. The particles do not solve differential equations. We do not know what the particles do. We just know that by setting up the differential equations we can ask questions about the motion of particles and get answers. We call this physics, and we forget that differential equations are a human construct, going back a surprisingly short time to Newton and Leibnitz, and that it is likely that the particles "know" how to move by obeying some other principle that we might or might not discover. (maybe quantum mechanics, maybe something else).

If this view is accepted, then chaos is not a limitation to science. Since we can discover what questions CAN be answered about chaotic sytems, we are not pertubed in the least that there exist questions that cannot be answered. We simply say "this is not physics", and go on in our busy schedule of progress in "understanding" nature.

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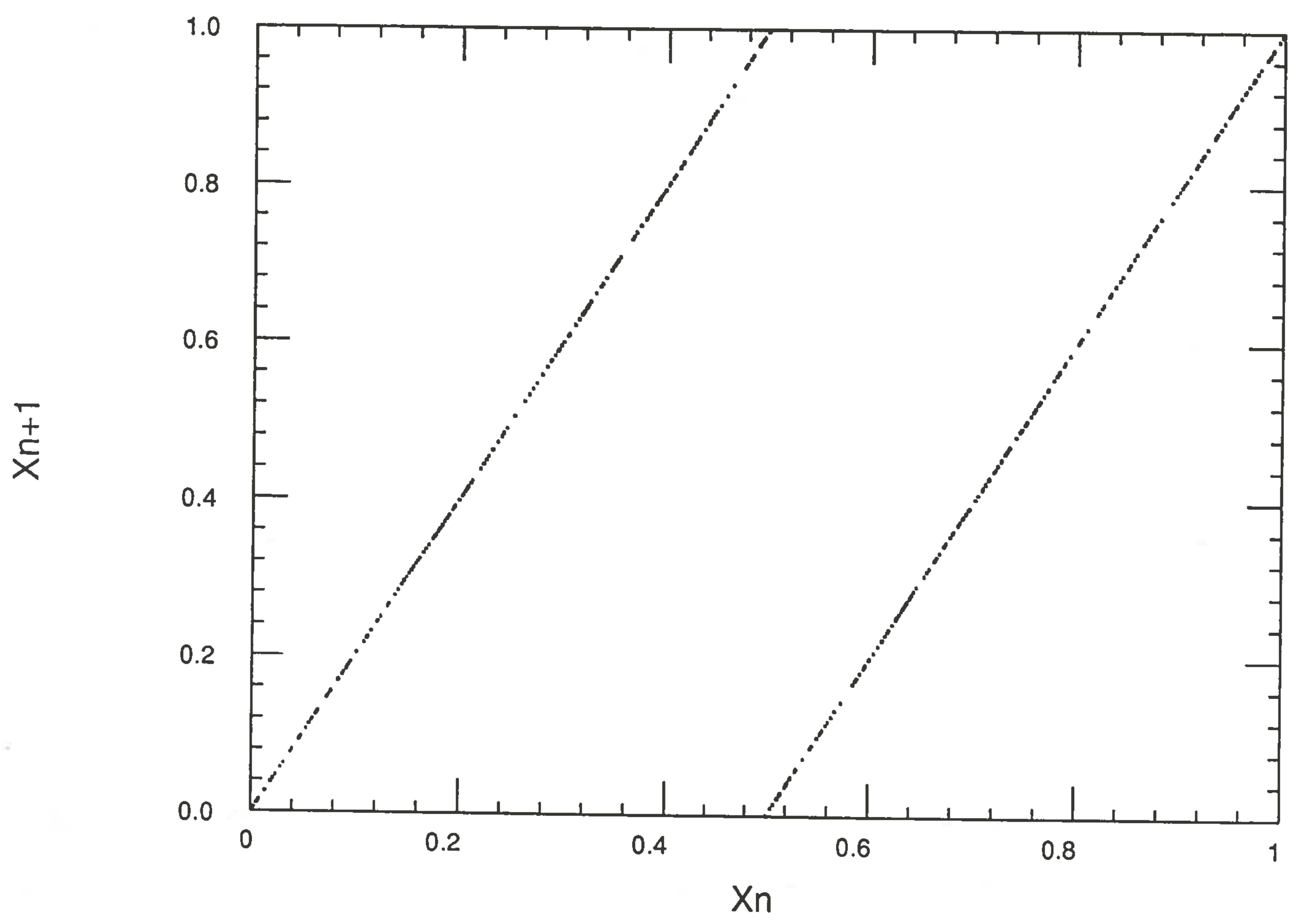

Fig 1a

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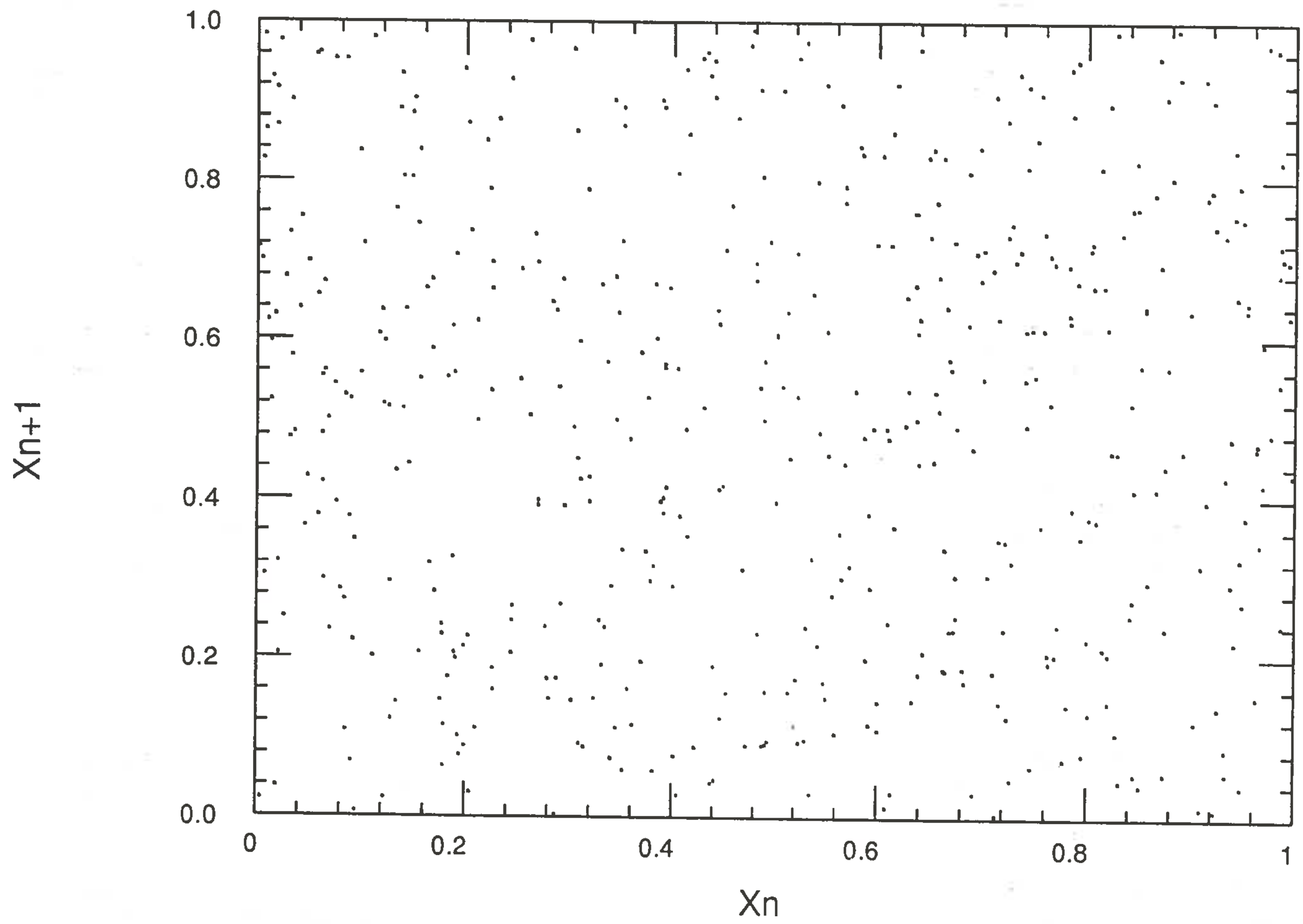

Fig 16

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