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Symmetry In Its Various Aspects:
Search for Order in the Universe

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HIDDEN SYMMETRIES IN TOPOLOGY AND HAMILTONIAN PHYSICS

by

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INTRODUCTION

Many problems of modern geometry and topology, mathematical physics and mechanics are reduced to the analysis of symmetries of corresponding differential equations. In cases when the group of symmetries is large, it is usually possible to integrate the differential equations (i.e. to find the solutions of physical problem) in "direct way". This is the reason why the problem of classification of all symmetry groups for dynamical systems is very important. But these symmetries usually are "hidden", and it is ~~hard~~ a difficult problem to find the symmetry group for concrete dynamical system. Recently the remarkable relation of this problem with topological bifurcation theory was discovered. It turns out that classification of dynamical systems which have "the maximal symmetry group" can be given in terms of one-dimensional and two-dimensional topological objects [1], [2], [3].

In the paper we illustrate this theory by visual material showing the hidden symmetries of concrete dynamical systems from classical mechanics.

1. IMPORTANT EXAMPLE: SYMMETRIES OF CLASSICAL EQUATIONS FOR THE MOTION OF A RIGID BODY IN 3-SPACE

The theory of the rigid body motion takes its origin from the classical works of Lagrange and Euler. The modern theory of motion of spacecrafts is also based to a considerable extent on the Euler-Poisson dynamic equations. Let Ω be the instantaneous angular velocity of the rigid body, and γ be the unit vertical vector in direction of z-axis for the fixed coordinate system in 3-space R^3 (Fig.1). We denote by M the vector of kinetic momentum of the body about point O. Let U be the potential function. Then, the well-known Euler-Poisson equations can be written in the form

$$dM/dt=M\times\Omega+(\partial U/\partial\gamma)\times\gamma, \quad d\gamma/dt=\gamma\times\Omega.$$

It was found long ago that the Euler-Poisson equations have three classical independent integrals: so called energy integral, area integral and geometrical integral. Thus, we can restrict the dynamical system on the 4-dimensional manifold given as common level surface of area integral and geometric integral. Consequently, for the complete integration of the equations, it is sufficient to find one more independent integral. The existence of this integral means that the system has 2-dimensional Abelian group of symmetries. In some sense integrability is equivalent to the "maximal symmetry property". In the Fig.2 well-known Lagrange top is shown - rotating rigid body with axial symmetry.

The question arises: which bodies (integrable or nonintegrable) are encountered more frequently? The intuitive, conceptual meaning of the question is clear: every rigid body (and hence its motion) is defined completely by its shape and initial conditions (at the beginning of the motion). A priori, the shape of the body is arbitrary. We call a body integrable (or "symmetrical") if its motion is integrable, say in Liouville sense, and nonintegrable ("non-symmetrical") otherwise. It turns out that if the shape of the body is chosen "by chance" or arbitrarily, its motion is "almost certainly" nonintegrable or random. It is clear intuitively that if a rigid body (like an asteroid or a bolid) has no symmetry, its motion looks like a random tumbling in 3-space. On the contrary, the shape of a rocket or spaceship is specially chosen to be symmetric to the highest possible extent to ensure the stability of its flight. This is due to the fact that, roughly speaking, integrability is a manifestation of symmetries in the shape of the body, while nonintegrability is associated with the lack of symmetry. Since the symmetry of a body is a "rare" phenomenon, while the typical case of general position is the absence of symmetries, nonintegrability or randomness of the motion of a body is a typical situation.

What is the behavior of a "typical nonintegrable system" on 4-dimensional phase-space? It was noted above that integrability (in the Liouville sense) indicates, among other things, that every 3-dimensional constant-energy surface Q defoliates into Liouville's tori and singular fibers. In particular, each integral trajectory of the general position remains all the time on "its own" 2-dimensional torus. On the contrary, if a system is nonintegrable, almost all integral trajectories in the typical case are found to move randomly over the 3-dimensional surface Q, each of them filling Q densely everywhere (such motions are sometimes called ergodic motions).

Let us now consider a "set of all Hamiltonian systems", actually the space of all smooth functions H (since the Hamiltonian dynamical system is specified by its Hamiltonian H). This space is infinite-dimensional. Which systems (integrable or nonintegrable, i.e. "symmetrical" or "nonsymmetrical") are encountered more frequently? It is clear intuitively that an integrable systems is a rare event in the myriad of all mathematically conceivable systems. At the same time, integrable cases are encountered more or less frequently in mathematical physics. It can be probably explained by the fact that the real physical Hamiltonians appear in the real world as they become "quite symmetric" for certain values of their parameters. At any rate, classical theoretical mechanics is based to a considerable extent on the ideas about a "certain harmony of the World" propounded in the works by Kepler, Copernicus, and many other outstanding scientists of Middle Ages. The concept of "harmony" in these works is based on the concept of symmetry. The vast body of experimental data accumulated by scientists over the last few centuries confirms to a certain extent the hypothesis according to which "symmetry reigns the world".

A question arises: is it possible to describe or classify integrable ("symmetrical") systems and isolate among them a "physical zone" comprising the systems which are encountered in real physics? The problem is very complicated, but in the last few years the new topological approach was discovered which allows to give the answer for "non-degenerate" (in some natural sense) Hamiltonian systems with two degrees of freedom.

One of the most vivid examples of physical "symmetrical" systems (i.e. integrable in Liouville sense) is associated with equations of motion of a heavy rigid body. We shall take the potential function U describing the ordinary gravitational field (gravity force field). The force of gravity is directed downwards along the vertical axis γ (Fig.3). It tuns out that the heavy rigid body having such a potential admits few important cases of integrability (i.e. "symmetry").

1) Euler's integrable ("symmetrical") case (was discovered in 1750). If the rigid body is fixed at its center of mass (viz. at point O, see Fig.4), i.e. $\rho=0$ (see $\rho=(r_1,r_2,r_3)$ on the Fig.3), such a Hamiltonian has another (additional) integral, so called Euler integral.

2) Lagrange's integrable case (was discovered in 1778). This case is sometimes called the symmetric top, or Lagrange's top. The symmetry of the system is manifested in the fact that the ellipsoid of inertia (with semiaxes A,B,C) is an ellipsoid of revolution (with two equal semiaxes) at point O (i.e., the fixed point about which the body rotates), and the center of gravity of the body lies on the rotational axis (Fig.2). A spinning top is a good model of Lagrange's top. The additional integral has a simple geometrical meaning: the component of the instantaneous angular velocity along the dynamic symmetry axis is conserved. Therefore, the symmetry of the system in this case is manifested clearly.

3) Kowalevskaya integrable case (was discovered in 1889). Here the integrable Hamiltonian H is defined by the following conditions: $A=B=2C$ and $r_3=0$. This case is more complicated to the previous cases since the second integral turned out to be not quadratic but fourth-degree polynomial. The symmetry of this integrable Hamiltonian is latent and associated with deep-rooted algebraic and geometrical properties of the Euler-Poisson equations.

2. CLASSIFICATION OF INTEGRABLE ("SYMMETRICAL") NONDEGENERATE HAMILTONIAN SYSTEMS WITH TWO DEGREES OF FREEDOM

1) Let us consider two integrable ("symmetrical") Hamiltonian systems of "general position". Are they orbitally

equivalent?

2) Classify all integrable systems to within the orbital equivalence. Does a topological invariant "responsible" for this classification exist?

3) Determine topological obstacles to integrability.

4) Describe all integrable systems of "low complexity".

5) Describe the "physical zone" in the table of all "mathematically existing" integrable systems.

It turns out that in a certain exact sense, all these problems can be solved. This is one of the results of the theory developed in the series of works by A.V.Bolsinov, A.T.Fomenko, H.Zieschang, S.V.Matveev, A.V.Brailov, A.A.Oschemkov, T.Z.Nguen, E.N.Selivanova, L.S.Polyakova, B.S.Kruglikov, O.E.Orel, V.S.Matveev, P.Topalov, V.V.Kalashnikov (junior), H.Dullin, S.Takahashi and others. See [1], [2], [3].

The classification is given in the terms of specific topological objects, which are called "atoms" and "molecules". We will describe here only these interesting objects and do not present the general classification theory. In turns out, that "atoms" and "molecules" became useful in many other geometrical problems and they describe the important topological bifurcations.

3. CODING OF MORSE FUNCTIONS ON TWO-SURFACES BY ATOMS AND MOLECULES

3.1. SIMPLE AND COMPLICATED MORSE FUNCTIONS

Let us consider a smooth function $f(x)$ on a smooth 2-dimensional surface (2-manifold) M. Let us recall that the function $f(x)$ is called Morse function, if its critical points are nondegenerate. The local structure of the level curves near the non-degenerate points is presented in the Fig.5. It is known that Morse functions are dense everywhere in the space of all smooth functions on a smooth manifold.

In other words, any smooth function can be converted into a Morse function as a result of even the slightest perturbation. In this case, complicated degenerate critical points are scattered to form a union of a certain number of the Morse-type (i.e., nondegenerate) singularities.

Henceforth, we shall denote by $f^{-1}(r)$ the complete inverse image (= preimage) of the value r of the function f. We shall denote by a the regular values of the function, i.e., the values whose preimage (inverse image) do not have a single critical points. In this case, $f^{-1}(a)$ is always a smooth curve in M.

We shall denote by c the critical values of the function, i.e., such values whose inverse image has at least one critical point.

Further, we can use the slightest perturbation to ensure that every critical level c (i.e., the set of points x for which $f(x)=c$) contains exactly one critical point. In other words, critical points falling on the same level can be moved to closely spaced levels (see Fig.6). Such Morse functions are sometimes called simple. The Morse functions which have several critical points (more that one) on at lest one critical level, we will call complicated.

Let us discuss the notions of simple and complicated Morse functions. As we recall above, each complicated Morse function can be deformed (by small perturbations) in a simple Morse functions. Of course, this fact is used in many applications and theoretical questions. From the other hand, there are important situations, when you cannot perturbate the complicated Morse function to transform it into a simple one. The first example is the theory of Morse functions with symmetries. If you are forced to investigate the properties of the Morse function (on some manifold) which is invariant under the action of some discrete group (group of symmetries), then, generally speaking, you cannot deform this function in a simple Morse function if you want to work in the same class of symmetrical Morse functions. We will discuss this problem in the next section. Now we will analyze the case of a simple Morse function. We will consider for simplicity the case of a Morse functions on a two-dimensional surfaces.

3.2. SIMPLE ATOMS AND CLASSICAL REEB GRAPH AS THE CODE OF A SIMPLE FUNCTION

a) REEB GRAPH

What is the structure of level curves of a simple Morse function defined on a two-dimensional surface M? Let us first consider an orientable surface.

If a is a regular value of the function, the relevant level curve consists of a few nonintersecting smooth circles (Fig.7). Let us replace each connected component of the total preimage $f^{-1}(r)$ by a single point. In the case when the value r=a is regular, each point will represent some smooth circle (lying inside the two-surface M). In the case when r=c is singular, the point can represent the single smooth circle or some complicated curve with singularities (which is one of the connected components of the level curve $f^{-1}(r)$) (Fig.8).

As a result, we obtain some graph (Fig.8). This graph is called Reeb graph for the Morse function. It is clear that this graph can be defined for any smooth function on the manifold M, i.e. not only for a simple Morse function. See example in Fig.9. It is evident that the Reeb graph allows us to reconstruct the evolution of the level curves of the function f on M for a simple Morse function. In this case each inner point on the edge of Reeb's graph represent some smooth circle and each vertex-point represent the figure of eight (Fig.8). But in the case of complicated Morse function the Reeb graph is too rough, because here it lost a lot of information about the behavior of the level curves of a function on the surface. The simple example you can see in the Fig.9, where two different functions (both are not a simple functions) have the same Reeb graphs W. For the function f_1 (which is complicated Morse function) the "saddle vertex" of the Reeb graph represents the curve with three critical points (Fig.10). For the function f_2 (which is even not a Morse function) the "saddle vertex" of the Reeb graph represents the singular curve with one complicated critical point (degenerated). It is easy to construct two different complicated Morse functions with the same Reeb graphs (Fig.11).

Thus, the Reeb graph is the nice code only for the case of a simple Morse function. Let us discuss its properties in more details for this special case.

b) MINIMUM AND MAXIMUM POINTS AND CORRESPONDING MINIMAX ATOM A

Let us consider a nonsingular level curve close to the minimum or the maximum of the function. This curve is homeomorphic to a circle. If the regular value tends to the minimax value (= minimum or maximum value), the circle shrinks to a point (Fig.12). In this case, the two-dimensional disk foliates into concentric circles with a common center (corresponding to the minimax value). We shall represent this evolution of level curves and the rearrangement by using the following conditional, but quite visual method. Every nonsingular level curve (circle) will be depicted by a point lying at the level a (Fig.12). Upon a change in a, this point will move and sweep a segment. At the moment when the value of the function becomes critical (equal to c), the circle will shrink to a point. We shall denote this event by the letter A with a segment emerging from it. The segment is directed downwards.

For the minimum, we proceed in exactly the same manner (Fig.12). In this case, the segment descends from above and terminates (at the lower end) at A.

We shall also assume that A denotes a disk with a point at the center, which foliates into concentric circles. We will call this surface with a boundary the minimax atom.

c) SADDLE POINT AND CORRESPONDING SADDLE ATOM B IN ORIENTABLE CASE

If c is the critical saddle-point value, the level curve has a figure-eight shape. When a tends to c, the two circles come closer and merge at a point where the level curve is rearranged. This process is also depicted in Fig.13. By reversing the direction of motion, we can speak of the inverse process, viz., the decomposition of a circle into two circles. The initial circle is "constricted", and then two circles stick together, after which the figure of eight thus obtained splits into two circles. Proceeding in the same way as in the case of minimax, i.e., presenting every regular circle by a point, and tracing their evolution (during the change in the level), we obtain a graph shown in Fig.13. This is a "tripod" oriented either upwards, or downwards. We shall denote the corresponding rearrangement by B, assuming at the same time that this letter also describes the pattern depicted in Fig.14, viz., a plane disk with two holes, which foliates into level curves of the Morse function. This surface will be called saddle atom B.

We will speak in this case about orientable saddle transformation (of the levels of a simple Morse function). It is evident that any simple Morse function on an orientable 2-surface has only orientable transformations (of its level curves).

d) REEB GRAPH AS MOLECULE FOR A SIMPLE MORSE FUNCTION IN ORIENTABLE CASE

The notation introduced by us is very convenient, for example, for solving the following problem. Let us suppose that a compact closed 2-surface is defined with a certain simple Morse function on it. Let all the critical points of this function be known. How can we reconstruct the surface? It runs out to be a simple problem in orientable case. We must consider all critical points of the function and the rearrangements corresponding to them, draw letters A and B on the relevant levels, and then connect the ends of the edges of these graphs. This leads to a certain graph W (Fig.15). Then we need to fix the orientation on all copies of atoms A and B and glue them using the diffeomorphisms of corresponding boundary circles which preserve the orientation. As a result we obtain a smooth closed 2-surface (without boundary if the graph W did not have a free ends). We will call this graph molecule (which is constructed from the atoms A and B).

It is clear that if the surface M is orientable, the graph W defines it unambiguously (up to diffeomorphism). It should be noted that the graph W is not necessarily plane graph.

It is important that the molecule W is considered as abstract graph without any fixed embedding in the Euclidean 3-space.

Thus, we have obtained a reasonable method to code a simple Morse functions of the orientable smooth 2-surfaces using the graphs W.

It is evident, that the Reeb's graph gives a nice code (model) only in the case of a simple Morse function. If the Morse function is complicated, then some vertices of a Reeb's graph correspond to a connected components of a singular level curves which contain several critical points. Thus, the topology of these level curves became very complicated and a single point=vertex of a Reeb's graph does not represent the complexity of this curve. So, we need in some new notion which can describe the topology of a level curves with several critical points. This notion was introduced for the purposes of topological classification of integrable Hamiltonian systems [1], [2].

Let us return to a simple Morse function on 2-surface.

Definition 1. Two-dimensional surfaces A and B (with a boundary) which foliate into circles as shown in Figs.12 and 14, will be referred to as atoms. Naturally, the graph W (Fig.15) will be called a molecule (since it consists of atoms).

An atom will also be denoted by a letter with a few positive and negative edges emerging from it (upwards and downwards) depending on the number of positive and negative boundary circles of the atom. Let us resume our discussion.

a) Any simple Morse function on an orientable 2-surface M can be coded by some graph (molecule) consisting of the atoms (vertices) of two types: A and B. All transformations of level curves of the function are orientable.

b) If some graph W without any free ends and consisting of the atoms of the types A and B is given, you can reconstruct in unique way (up to diffeomorphism) some compact orientable 2-surface with some simple Morse function on it such that the corresponding molecule will coincide with a given graph W.

c) If two molecules W and W' are homeomorphic, then corresponding 2-surfaces M and M' are diffeomorphic.

Thus, the molecule W is a reasonable code for the pair (M^2, f), where M is a surface, f is a Morse function. It is clear that the same 2-surface M can carry different simple Morse function. Consequently, different molecules W and W' can represent the same 2-surface (but different Morse functions on it).

e) SADDLE ATOMS B AND A^* FOR A SIMPLE MORSE FUNCTION IN NONORIENTABLE CASE

We now reject the hypothesis of the orientability of the surface M, i.e., go over to the general case. The minimax rearrangements of the type A have the same construction both in the orientable and nonorientable cases. The difference appears in the case of a saddle point. Let us first recollect how a saddle-point - type rearrangement (with an index equal to unity) actually takes place in the orientable case. It is depicted in Fig.16. A narrow strip (rectangle) is glued to the pair of boundary circles (representing the boundary of the manifold: $f(x) \leqslant c-\varepsilon$, where ε is a small quantity). The gluing is such that the obtained surface remains orientable. As a result, the boundary is found to be homeomorphic to a circle.

Let us now consider the case when a rearrangement takes place within a nonorientable surface. Some rearrangements (with an index 1, i.e., of the saddle-point type) can occur in this case as for an orientable surface. However, here can appear the new rearrangement which is made according to a completely different principle. This rearrangement is shown in Fig.17. A twisted (by 180°) rectangle is glued to the same boundary circle of the surface. As a result, there appears a new Möbius strip within the surface $f \leqslant c+\varepsilon$. Clearly, there remains only one boundary circle after the rearrangement. Thus, as we go over through the critical level c, a circle is transformed again into a circle. Using the symbols introduced by us earlier, i.e. depicting every nonsingular connected level curve (viz., a circle) by a point, we must represent the evolution described above as shown in Fig.17; the edge of the graph with the letter A^* at the middle. This letter denotes conditionally a "nonorientable" rearrangement.

Thus, if f is a simple Morse function on a compact closed surface (which is orientable or nonorientable), we can put it in correspondence with the graph W having vertices of the type A, B, or A^*.

What are the specific features of the rearrangement of circles upon a transition through the critical level in the case of A^*? Figure 18 shows the surface $f^{-1}(c+\varepsilon,c-\varepsilon)$. It has the form of two Möbius strips glued crosswise.

The "minus" sign marks the circle lying on the level $c-\varepsilon$ while the "plus" sign shows the circle on the level $c+\varepsilon$. The critical saddle point is at the center.

Let us define a new atom A^* corresponding to the rearrangement. In the orientable case, we simply took the strips $P_c=f^{-1}(c+\varepsilon,c-\varepsilon)$ for atoms A and B. Here we also will take the 2-surface $P_c=f^{-1}(c+\varepsilon,c-\varepsilon)$ and will call it atom.

The atom P_c is called orientable if its 2-surface P_c is orientable. The atom P_c is called non-orientable if P_c is non-orientable.

So, atom B is orientable and atom A^* is non-orientable.

The atom A has only one boundary circle, and hence only one edge emerges from the vertex A. Accordingly, two edges emerge from the vertex A^* and three edges from the vertex B.

3.3. EXAMPLES OF SIMPLE MORSE FUNCTIONS ON NONORIENTABLE SURFACES: PROJECTIVE PLANE AND KLEIN BOTTLE

a) PROJECTIVE PLANE

Let us recall that projective plane RP^2 can be represented as 2-surface obtained by gluing of the square according the rule shown in the Fig.19. We need to identify the edges of the square in such a way that two copies of the letter a are glued (with their orientations) and two copies of the letter b also are glued (with identification of the arrows). Another evident model: the two-dimensional disc with identified opposite points on its boundary circle. The evident equivalent model: two-sphere with a hole with identified opposite points on the boundary of the hole.

It is easy to see that projective plane can be obtained from Möbius strip if we glue μ with 2-disc D^2 by identification of their boundary circles, i.e. $RP^2=\mu+D^2$. In other words, if we remove 2-disc D^2 from projective plane RP^2, we obtain Möbius strip μ, i.e. $RP^2-D^2=\mu$.

Let us consider the simple Morse function f which is determined on projective plane by the set of its level curves shown in the Fig.20. It is clear that there are many smooth functions with the same collection of level curves, but among these functions there are Morse functions and they are simple. Let us analyze this Morse function in details. Its minimum point is marked in the Fig.20 as m_-, its maximum point is m_+ and its saddle point is S (in the center of the square). There is a slightly different representation of this function shown in the right side of the Fig.20. The critical saddle level curve is obtained from two diagonals of the square by identification of their ends marked as P and Q. Let us note that these two points P and Q are different on the RP^2 (after gluing of the square's boundary). Thus, critical saddle level is figure of eight.

Let us consider the following decomposition of projective plane: $RP^2=D^2+A^*+D^2$. Here first D^2 is the neighborhood of the minimum point m_-, the second D^2 is the neighborhood of the maximum point m_+ and A^* is the tubular neighborhood of the saddle critical level curve. It follows from the Fig.21 that A^* is homeomorphic to the same surface which was denoted as A^* in the previous section and is obtained from the "cross" by identification of its ends according the rule shown in the Fig.251.

Finally, this simple Morse function is represented by the molecule W shown in the Fig.22.

b) KLEIN BOTTLE

Klein bottle is the 2-surface which is obtained from square by identification of its edges according the rule shown in the Fig.23. It is easy to see that Klein bottle KL^2 is obtained from two Möbius strips by gluing their boundary circles. So, we can write that $KL^2=\mu+\mu$. We can use this decomposition to construct the simple Morse function on Klein bottle. This function is determined (non-uniquely) by its set of level curves shown in the Fig.24. Here we have one minimum point m_-, one maximum point m_+ and two non-degenerate saddle points R and S. Each saddle critical level curve is homeomorphic to the figure of eight. Each saddle point correspond to the atom A^* and consequently, the whole Morse function is described by the molecule

$$A —— A^* —— A^* —— A.$$

3.4. COMPLICATED MORSE FUNCTIONS, COMPLICATED ATOMS AND GENERAL MOLECULES

a) EXAMPLE: FUNCTIONS WITH SYMMETRIES AS COMPLICATED MORSE FUNCTIONS

Let us consider some smooth manifold M and a Morse function f(x) on M. Assume that this function is invariant under the action of some discrete group G on M. This means that each element g of the group G is represented by some diffeomorphism of the manifold M (we will denote this diffeomorphism by the same letter g:M--->M), and the function f satisfies to the following condition: f(g(x))=f(x) for every point x from M. In other words, the function f is constant on every orbit of the group G in X. The orbit G(x) of the point x is the set of all points in M which have the form g(x) (where g runs through the whole group G), i.e. - the images of the point x under the shifts by the diffeomorphisms g from the group G. Such functions f are called usually the functions with symmetries (where the group G is called the group of symmetries).

Let us consider the simple example in the Fig.25: the height function f(x) which is defined on the flat ring (2-surface with boundary) and has five maxima and five saddle critical points (so, totally this function has ten critical points). Here the surface M is the ring and the group G is the group Z_5 - the cyclic Abelian group of the order 5. This group is the group of orthogonal rotations about the vertical z-axis in Euclidean 3-space. The generator of this group is the rotation on the angle $2\pi/5$. Such functions appear in many applied problems of modern geometry and topology.

Let us consider the general Morse function f on M which is invariant under the action of the group G. Let x_0 be a critical point of this function. Then we state that all points $g(x_0)$ (where g runs trough the whole group G) also are the critical points of the function f (with the same index). Moreover, the function f takes in all these points the value equals to the value $f(x_0)$. It follows immediately from the definition of the invariant function and from the remark that this function is always constant on the orbit.

Consequently, if the orbit of the critical point x_0 does not coincide with the point x_0, then the critical level $f^{-1}(c)$, where $c = f(x_0)$ contains several critical points (which are the images of the critical point x_0). Thus, such function f is the complicated Morse function.

Resume. Let us assume that f(x) is a Morse function on a smooth manifold M invariant under the action of some discrete group $\mathfrak{G}$ and assume that there exists at least one critical point of f which is not a fixed point of the action of G. Then the function f is a complicated Morse function.

In the example in the Fig.25 the saddle critical level of the complicated Morse function contains five critical saddle points. The critical level curve is obtained by gluing of five circles (each two heighbouring circles are glued in one point). The maximal critical level curve consists of five isolated maximal points.

It is clear that if we have the complicated Morse function which is realized as the function with symmetries (on M), then in general case we cannot transform this function (by a small perturbation) in a simple Morse function if we want to preserve its invariance. It is clear shown in the Fig.26. We see the small perturbation of the initial function f in the function $\tilde{f}$ which is close to f. The critical (singular) level curve of f containing the five saddle points is shown. The small perturbation transforms this curve in the union of several figure-eight curves. It is evident that we lost the invariance property of the function: the new function $\tilde{f}$ is not invariant under the action of the group Z_5.

Thus, if we want to investigate the properties of invariant (symmetrical) Morse functions, we are forced to consider a complicated Morse functions. We also are forced to introduce in the theory of coding some new objects which will allow us to describe the complicated critical level curves with many critical points (on the same level).

b) COMPLICATED MORSE FUNCTIONS, COMPLICATED ATOMS AND COMPLICATED MOLECULES. ORIENTABLE CASE

The constructions described above for the simple Morse functions can be extended to the complicated Morse functions. The only (but very important difference) is that the number of atoms increases, and the atoms become more complicated.

Let M be a closed two-dimensional smooth surface (orientable or nonorientable).

Let f be a Morse function of the general type, for which there can be several critical points on a critical level.

If c is the minimax critical value of the function f, the rearrangement of a nonsingular level curve (consisting of several nonintersecting circles) occurs as in the case of a simple Morse function, i.e., consists of a few type A rearrangements.

Let us first assume that the surface M is orientable.

Definition 2. We shall call a connected component of the 2-surface $f^{-1}(c-\varepsilon,c+\varepsilon)$ an atom corresponding to the critical value c (there can be a few atoms corresponding to c if they lie on the same level of the function). We shall denote the atom by P_c^2 (or P_c). The surface P_c is always orientable because the surface M is supposed to be orientable. The critical points of the function f lying on P_c^2 are called vertices of the atom. In the case when the surface M is orientable, these vertices can only have a multiplicity of 0 (the isolated minimax point of the function), or 4 (the saddle point is at the center of the "cross"; four edges converge to this vertex).

The surface P_c^2 has a boundary consisting of a certain number of circles. The circles lying at the level $c-\varepsilon$ will be called negative, while those lying at the level $c+\varepsilon$ will be referred to as positive. Their number can be different. A singular level curve, viz., the connected component of the curve $f^{-1}(c)$ (which will be denoted by K_c) is a connected closed curve with singularities. The singularities of K_c are exactly the vertices of the atom. The graph K_c can be naturally called the skeleton, or spine of the atom (see example in Fig.27).

It should be noted that different atoms may have the same spine K_c. Therefore, K_c itself does not define the atom uniquely. The surface P_c obviously contracts to its own spine. A spine is the deformational retract of its atom (i.e., the spine remains stationary all the time when the atom contracts to it). This contraction can be carried out along the gradient lines of the function f defined on the atom.

Let us describe some properties of atoms. If we omit the graph K_c from the atom, the latter will disintegrate into a union of a few rings. Near every edge of the graph K_c there is exactly one positive and exactly one negative circle. The vertices of the atom, viz., the singular points (vertices) of the graph K_c, may only have a multiplicity of 0 or 4. The graph K_c cannot be a "pure" circle without vertices since by definition it is a singular level curve of the function f passing through critical points (at least through one). The critical points are just the vertices of the graph. By the way, we could complete the picture by including a simple circle (loop without vertices) into list of graphs K_c by considering the connected component of a regular level curve of the function f.

Figure 27 shows an example of an atom which will be denoted by D_1 (the meaning of this notation is not now important for us, but it is reasonable to mention that these notations appeared as a reflection of Hamiltonian physics, where the theory of atoms and molecules is used, see [1]). This atom D_1 is a plane surface (i.e., can be realized in the form of a domain on a plane). However, not all atoms are plane. Am example of a different atom is shown in Fig.28. We can easily construct a Morse function realizing this atom. Such an atom cannot be embedded into a plane. However, it can be immersed in the plane by allowing self-intersections (superpositions).

Clearly, the number of atoms is infinitely large. On the other hand, they can be easily ordered and classified as their complexity increases. Let us describe a useful and graphical method of depicting atoms. It is known from topology that any two-dimensional orientable surface with a boundary can be immersed in a two-dimensional sphere. Therefore, any atom can be visualized as immersed in a sphere. Naturally, we shall not distinguish between the immersions of an atom which can be obtained from one another by a smooth deformation (isotopy) within the sphere. Besides, we agree not to distinguish between immersions differing from one another in the "loops" shown in Fig.29. By extruding a point (which does not lie on the atom) from the sphere, we can depict the atom immersed in the plane.

The picture of an atom can be simplified still further. Indeed, we shall define the atom completely (and uniquely) by specifying only the immersions in the sphere of its graph K_c. If the immersion of the graph is specified, it is sufficient to consider a small tubular neighborhood of this immersion. This neighborhood is nothing but the atom (2-surface with a boundary). This idea is illustrated in Fig.29 for the atom C_1.

Conclusion. An atom is the immersion of the graph K_c in a sphere S^2 (or a plane R^2 if we extrude from sphere a point not lying on the graph). Isotopic immersions of the graph and the immersions differing only in loops are assumed to be identical (to be more precise, equivalent).

Therefore, while depicting henceforth the atoms P_c, we shall consider only their "skeletons" K_c immersed in a plane. Figure 29 also shows examples of equivalent and nonequivalent immersions (in a sphere) of the same graph K_c.

Thus, the set of all orientable atoms forms an infinitely long discrete list. We shall define the complexity of an atom as the number of its vertices. It should be recalled that the vertices of atoms can have a multiplicity of 0 (isolated vertex) or 4 (cross). There exists an algorithm (realized on a computer) which enumerate consecutively all atoms in the order of increasing complexity. We shall present here only the beginning of this list (see Fig.30), containing atoms of complexity 1, 2, and 3 (specified by immersions of the graphs K_c in a 2-sphere). In the Fig.31-a and Fig.31-b we represent the same atoms in the form of 2-surfaces with boundary.

As in the case considered above, we can put a molecule W in correspondence with a Morse function on the surface M by connecting the corresponding atoms by edges. As a result, we obtain the graph W with atoms as its vertices.

In conclusion let us shown an interesting example of complicated Morse function on two-dimensional torus. The torus is obtained from flat square (Fig.32) by gluing its edges according the rule given by the arrows and letters on the edges. Let us consider the function f on the torus T^2 determined (non-uniquely) by its level curves shown in the Fig.33. This function has two non-degenerate saddle critical points, one minimum point and one maximum point. The saddle atom is homeomorphic to the surface shown in the Fig.33 and is denoted as C_1 (according to our classification list).

Resume. The atoms represent and classify the different types of symmetries appearing in the theory of complicated Morse functions on 2-surfaces. From the other hand, the same atoms represent and classify the different types on bifurcations in integrable ("symmetrical") Hamiltonian systems [1], [2] ,[3].

4. TOPOLOGICAL ATOMS AND MOLECULES CLASSIFY THE BIFURCATIONS AND SYMMETRY TYPES OF INTEGRABLE SYSTEMS

Topological 2-atoms desribed above are represented as 2-dimensional surfaces with boundary, foliated by level curves of

Morse function. According to definition, 2-atom represents the bifurcation of level curves of Morse function when the value of the function crosses the critical value. Let us consider the direct product of 2-atom on the circle. We obtain some 3-dimensional manifold with boundary. We call this product 3-atom. This manifold is foliated by 2-surfaces which are the direct products of the level curves on the cirle. Almost all of these 2-surfaces are 2-dimensional tori. But there is one isolated singular fiber which is more complicated that 2-torus.

The first example is shown in the Fig.34, where we take the direct product of the atom A on the cirle. We obtain the foliation of 3-dimensional solid torus by concentric 2-tori which are contracted on the circle.

Second example we obtain by multiplication of the atom B on the circle (Fig.35). Here two tori are transformed into one torus after passing trough the critical (singular) level.

More complicated example ise shown in the Fig.36. Here we consider so called twisted Seifert product of the atom B on the circle.

One of the main theorems of the theory claims that all possible orientable bifurcations of nondegenerate integrable Hamiltonian systems with two degrees of freedom are classified be the set of all 3-atoms. In this sense the infinite list of all 3-atoms (or in equivalent way the list of all 2-atoms) represents the list of all possible orientable "transformations of symmetry" in integrable systems.

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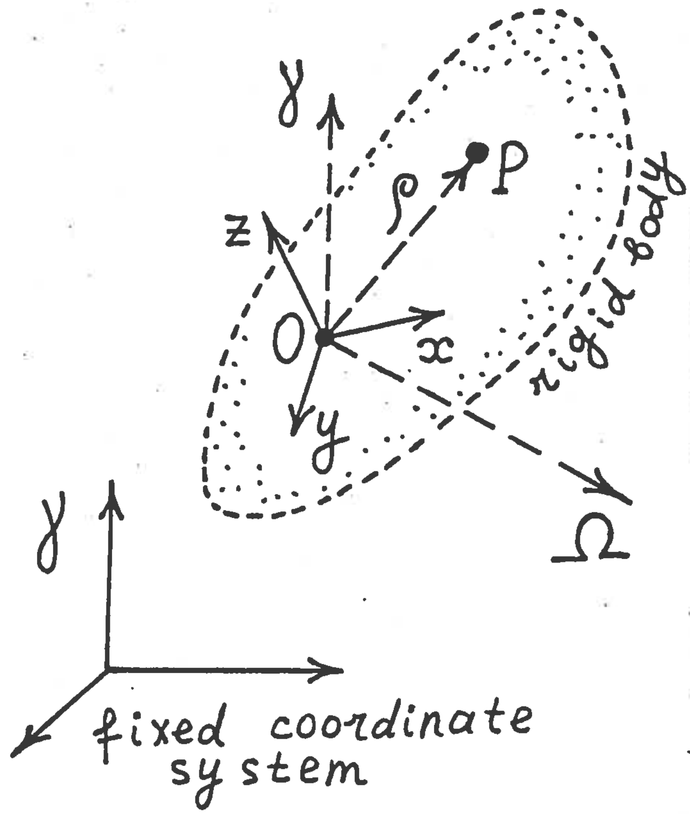


Fig. 1

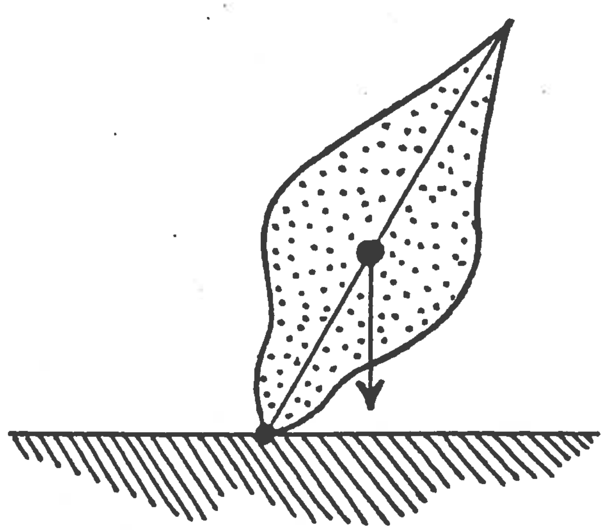


Fig. 2

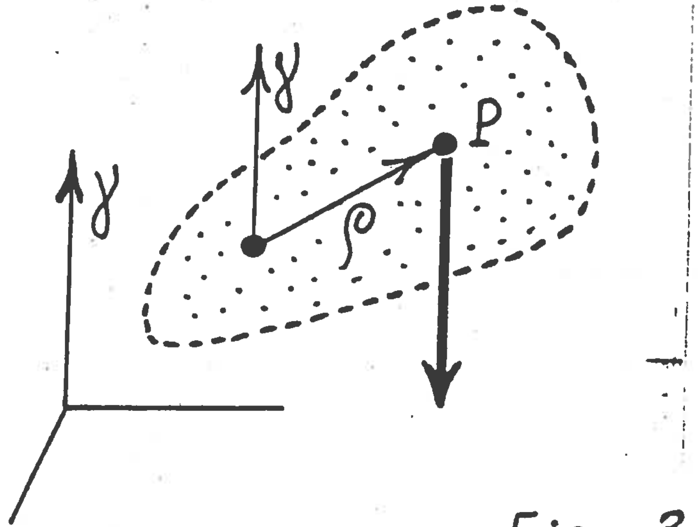


Fig. 3

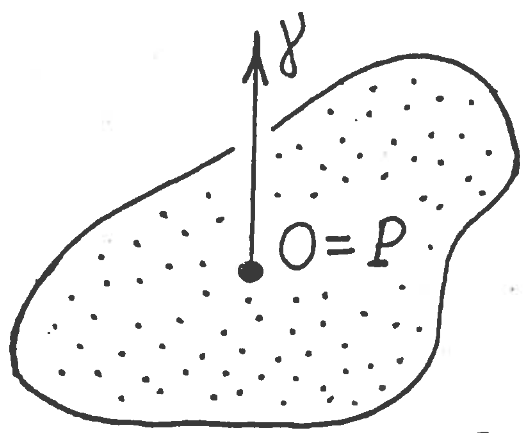


Fig. 4

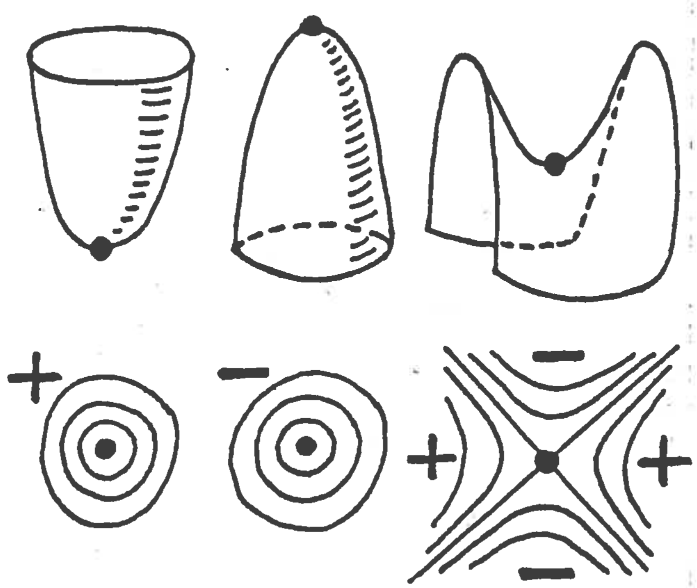


Fig. 5

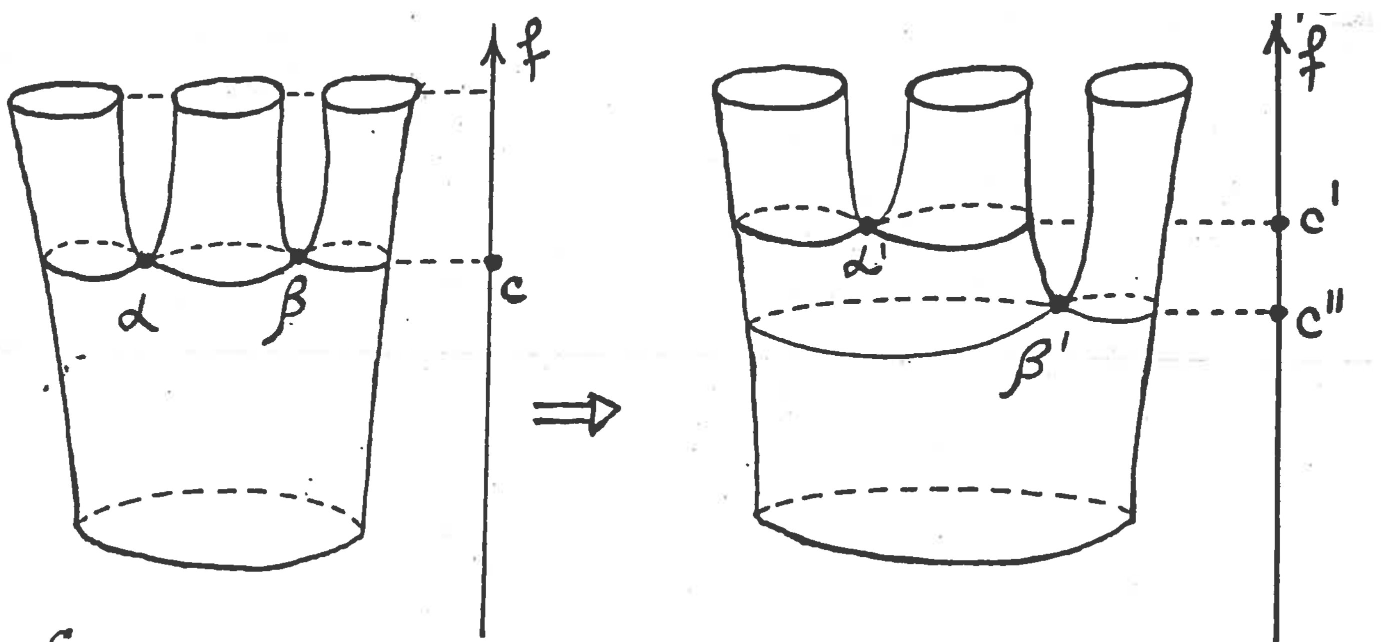


Рис. 6

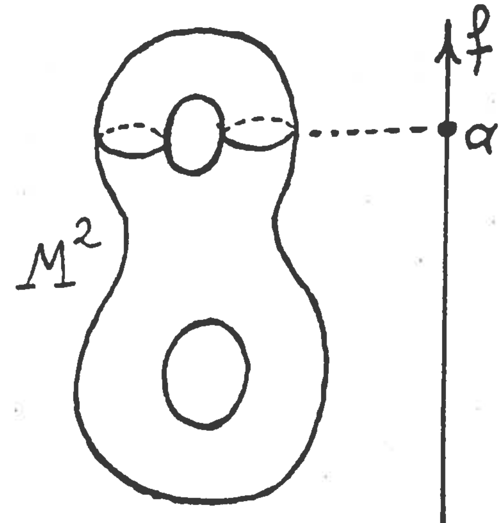


Fig. 7

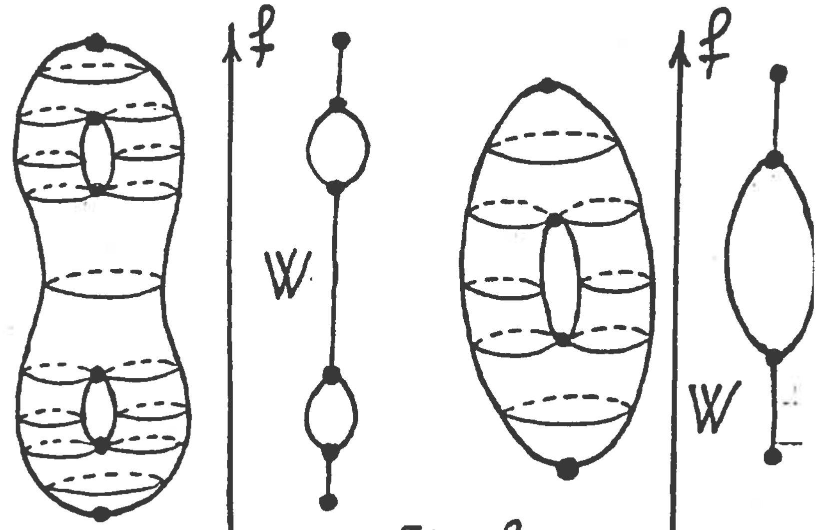


Fig. 8

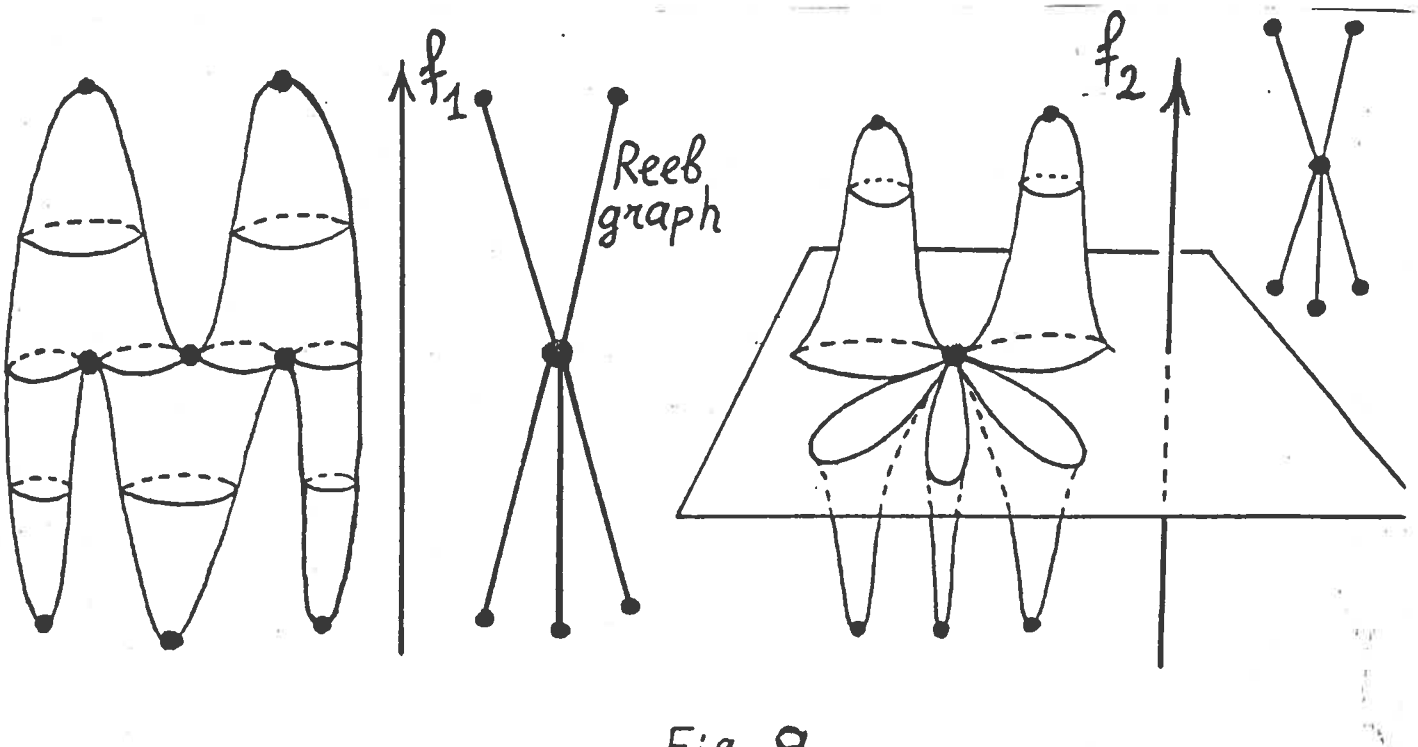
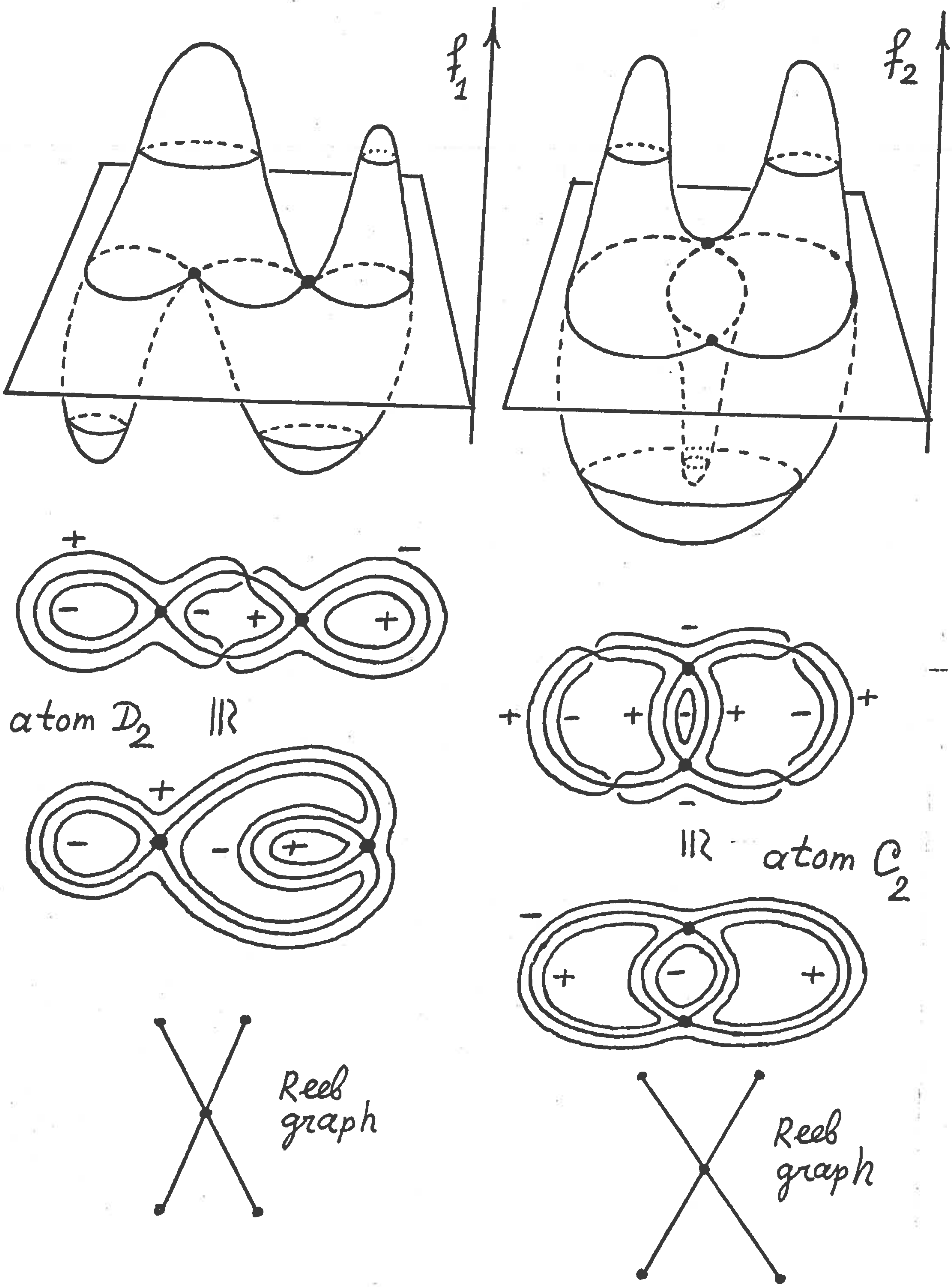


Fig. 9



f_1
f_2
atom D_2
atom C_2
Reeb graph
Reeb graph

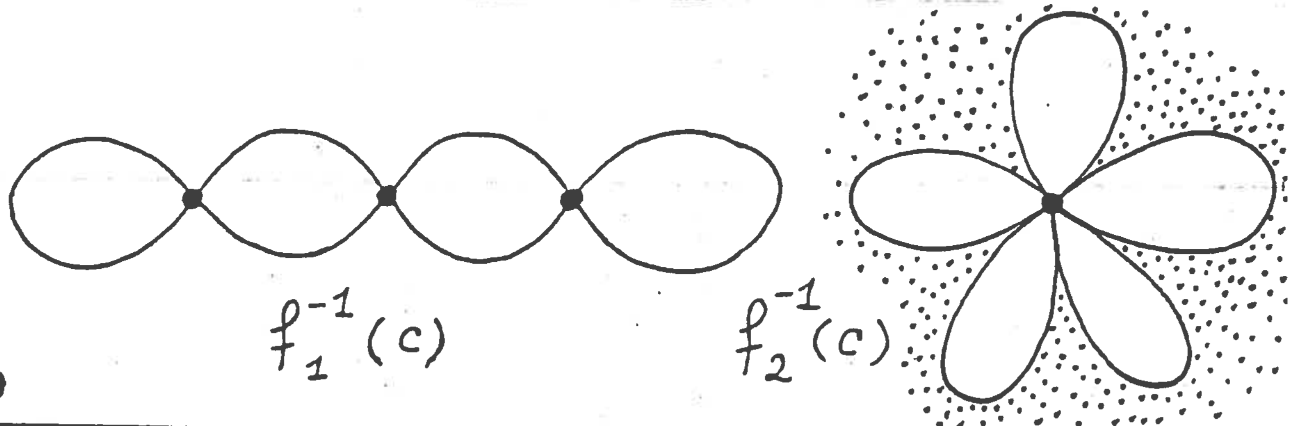


Fig. 10

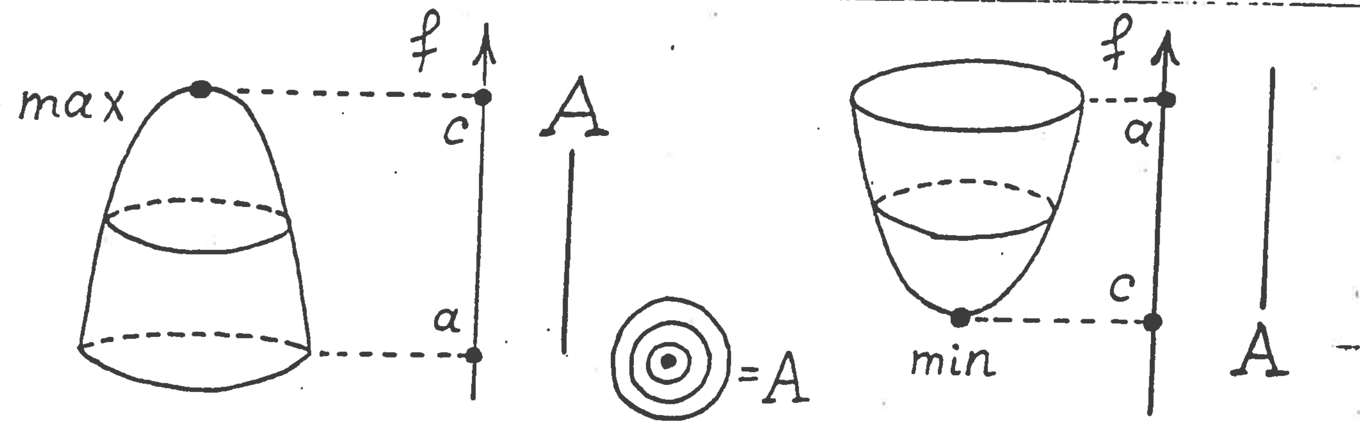


Fig. 12

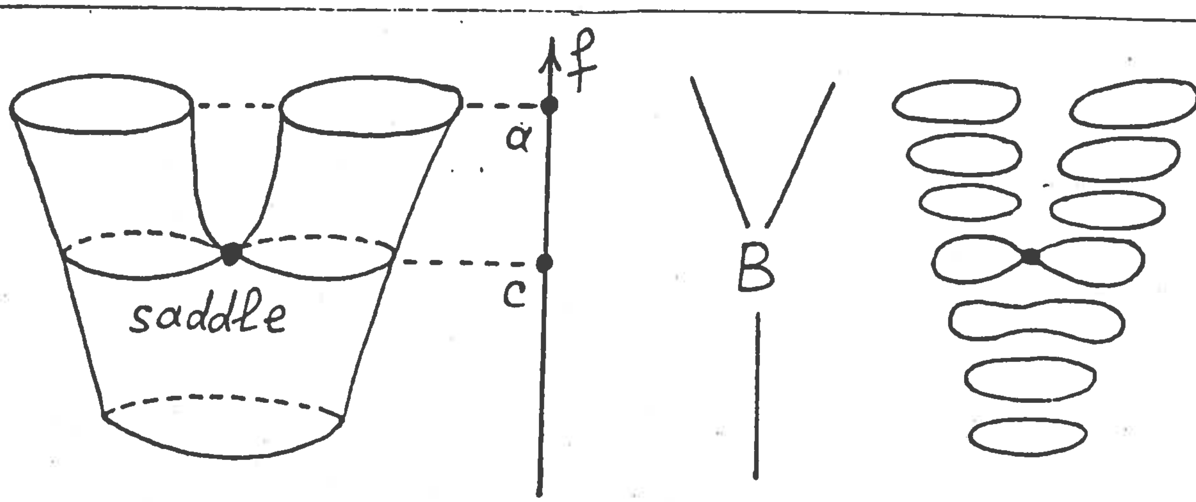


Fig. 13

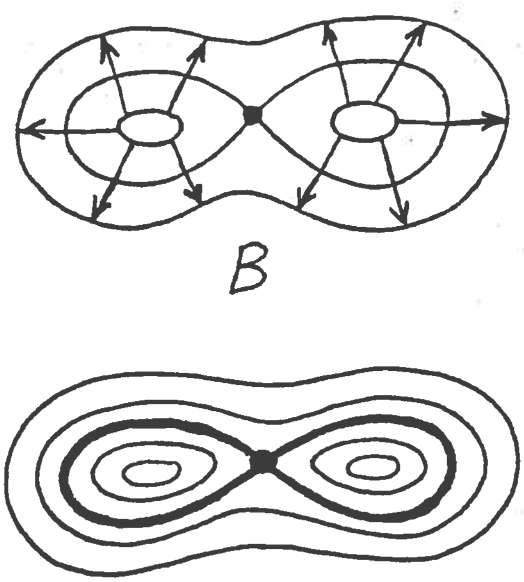


Fig. 14

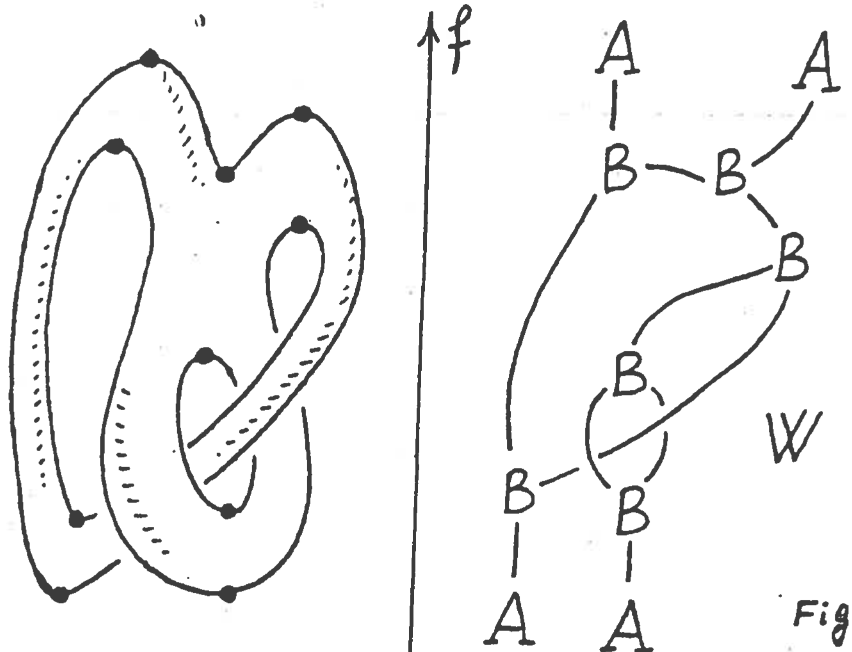


Fig 15

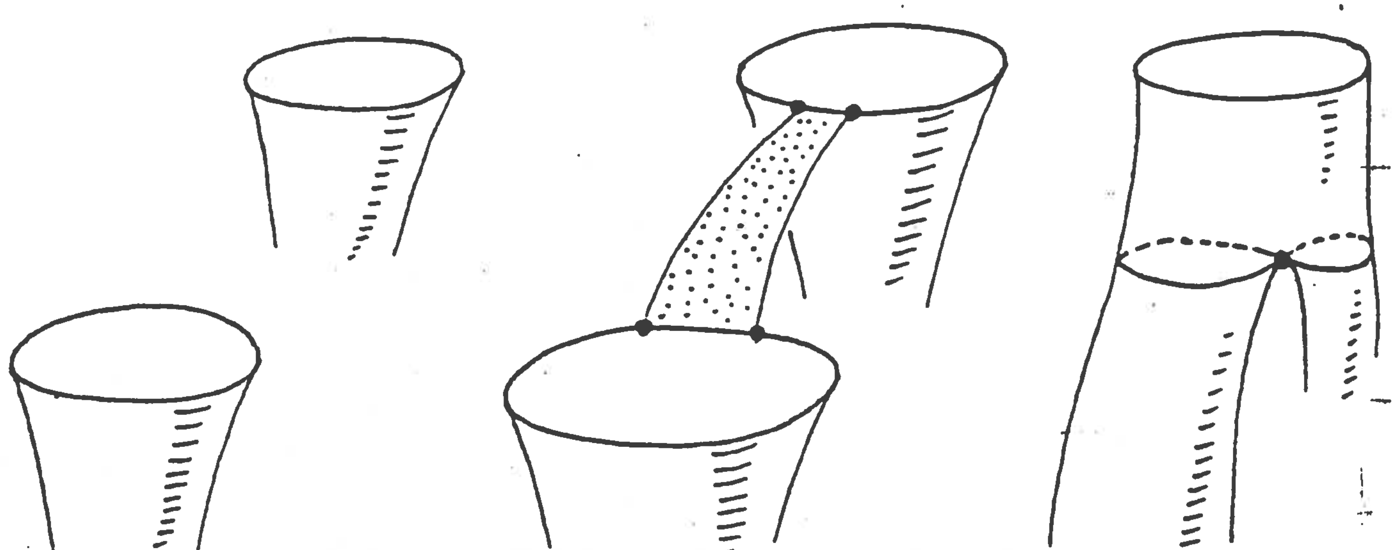


Fig. 16

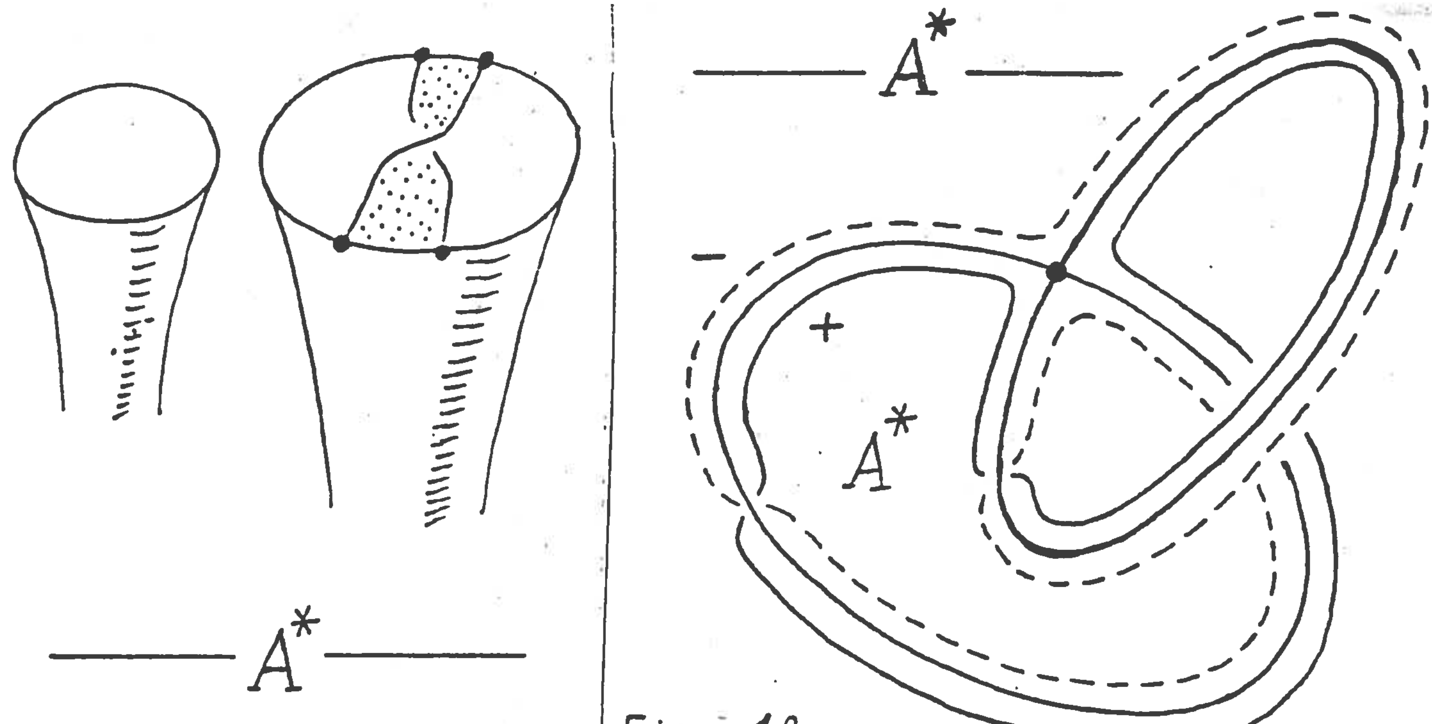
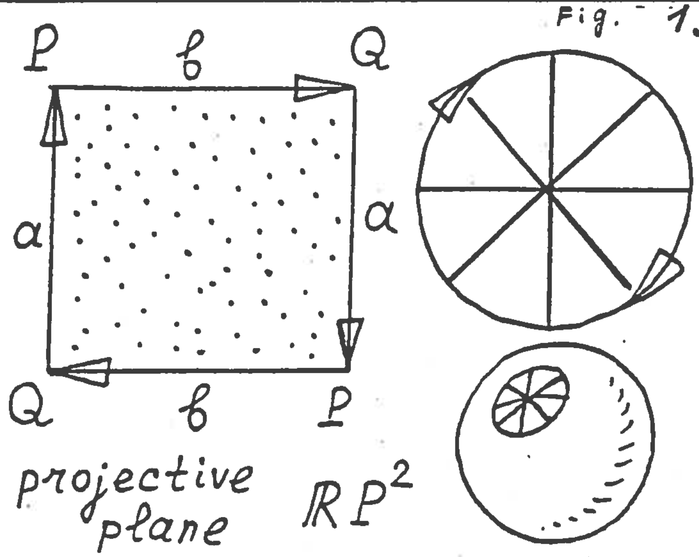


Fig. 18

Fig. 19



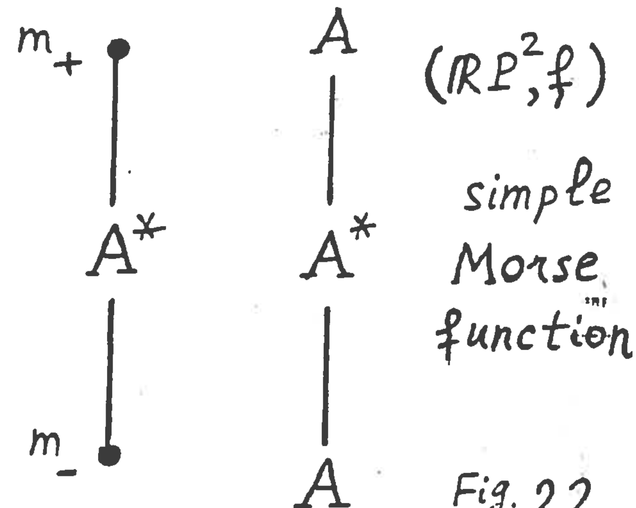
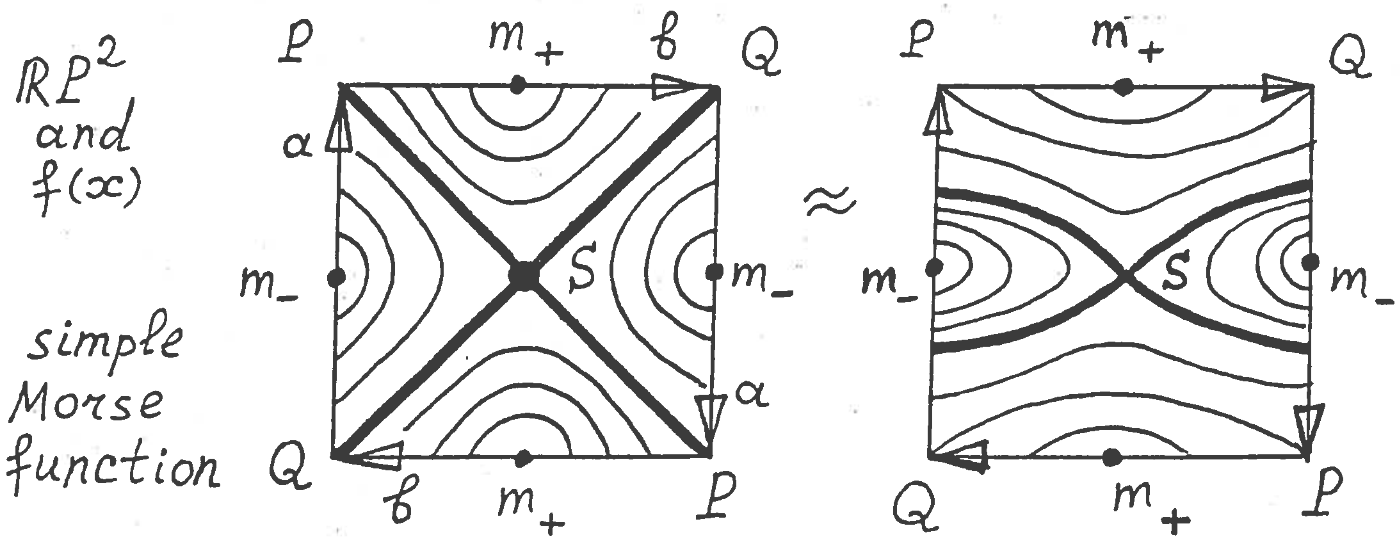


Fig. 22



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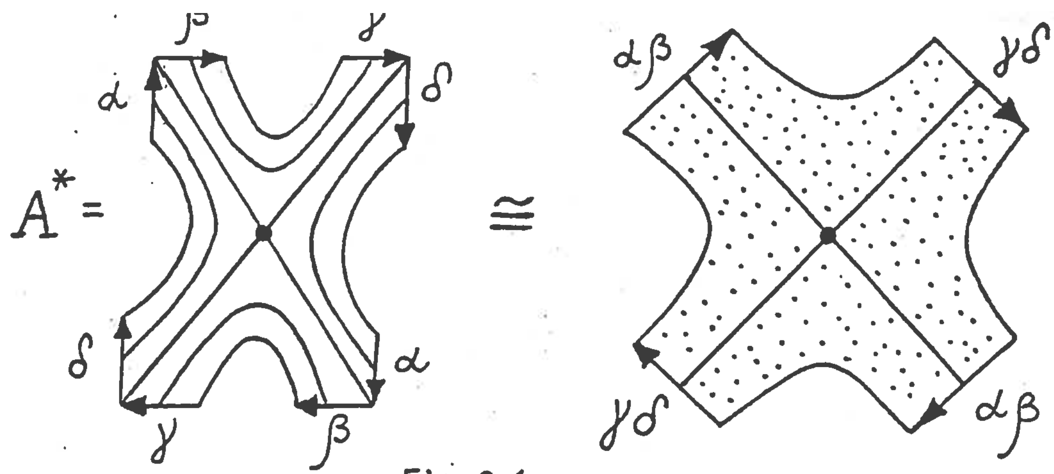

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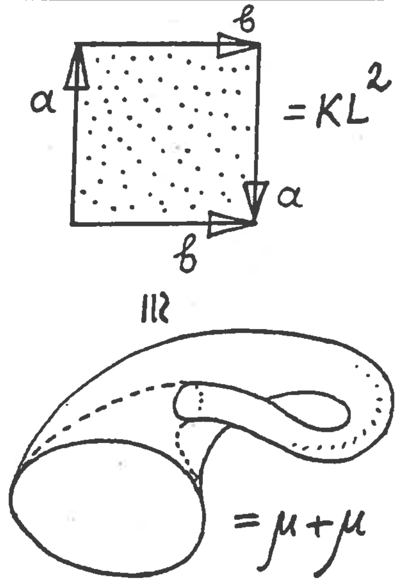
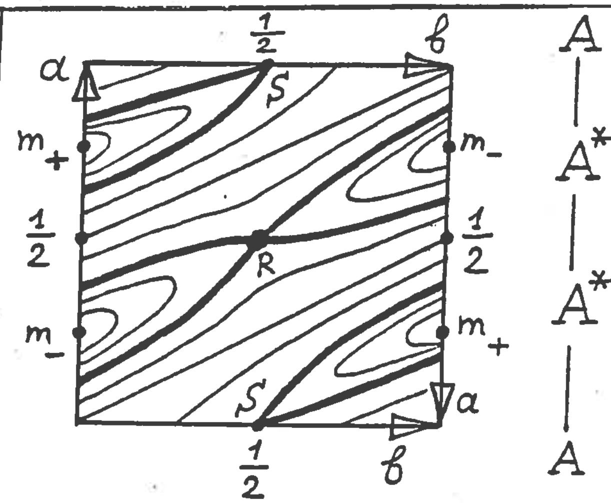


Fig. 23



simple Morse function on Klein bottle

Fig. 24

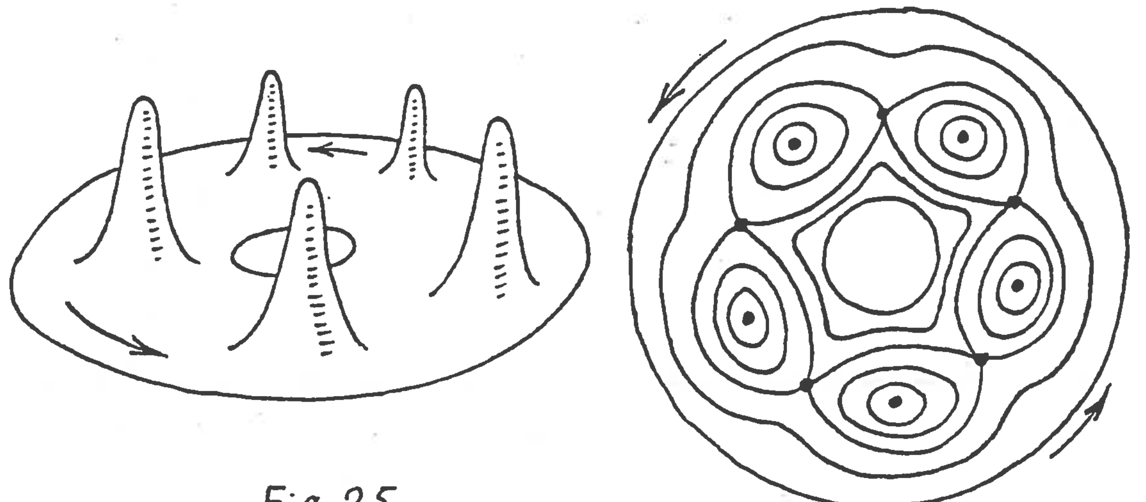


Fig. 25

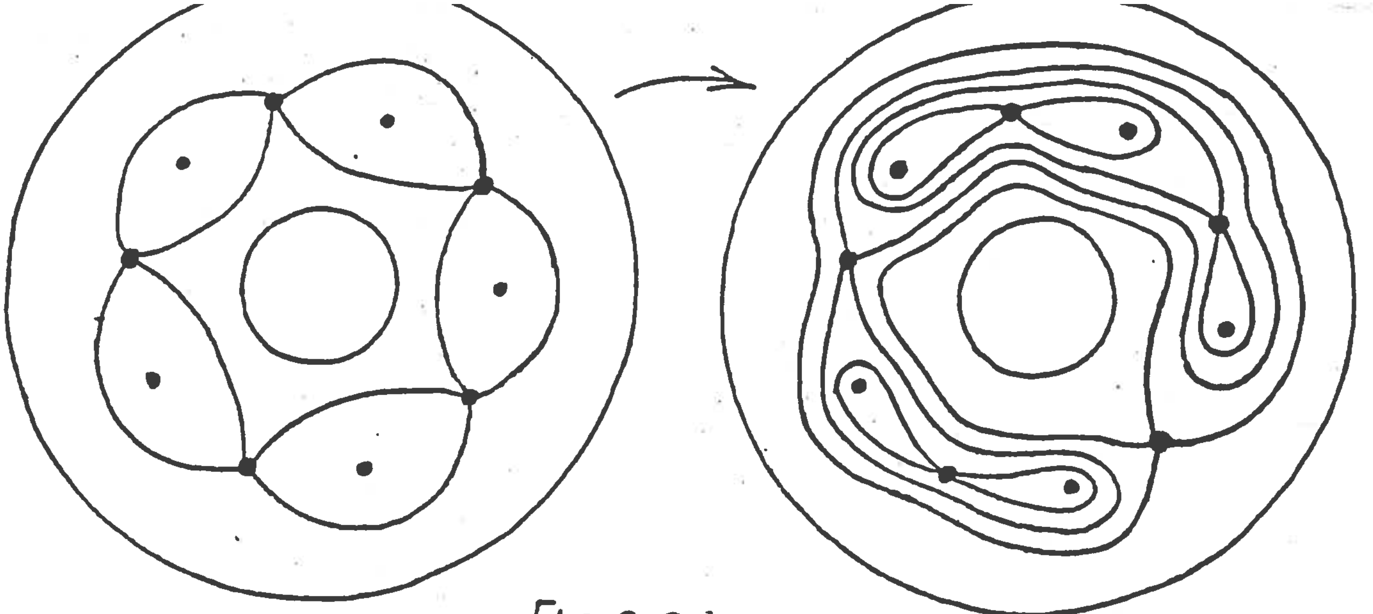


Fig. 26

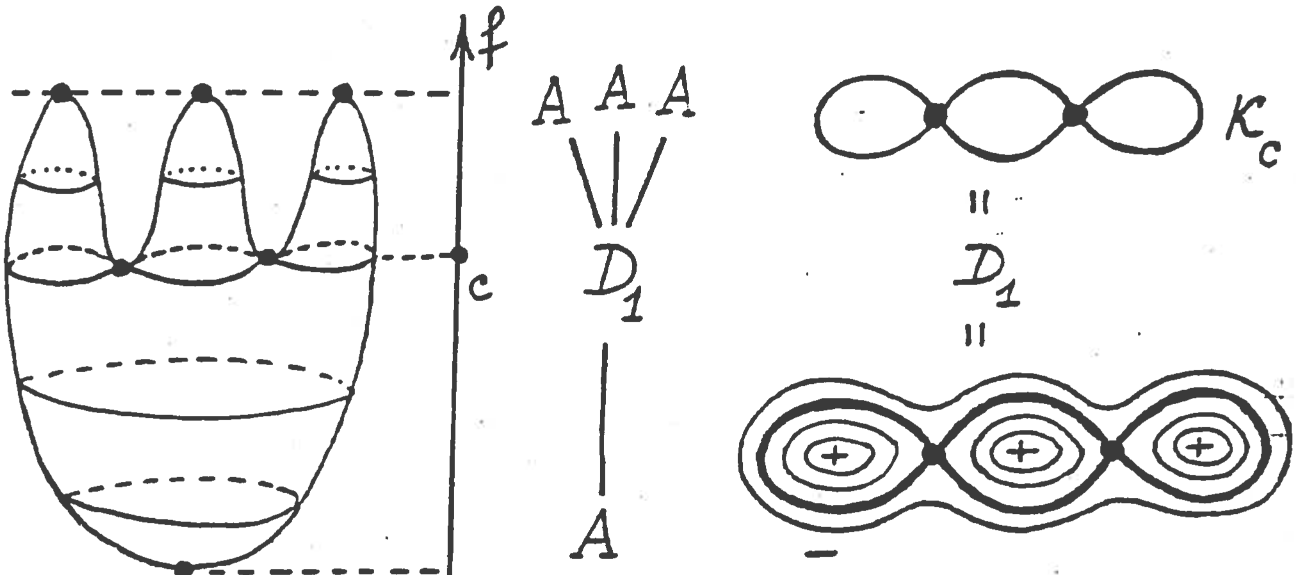


Fig. 27

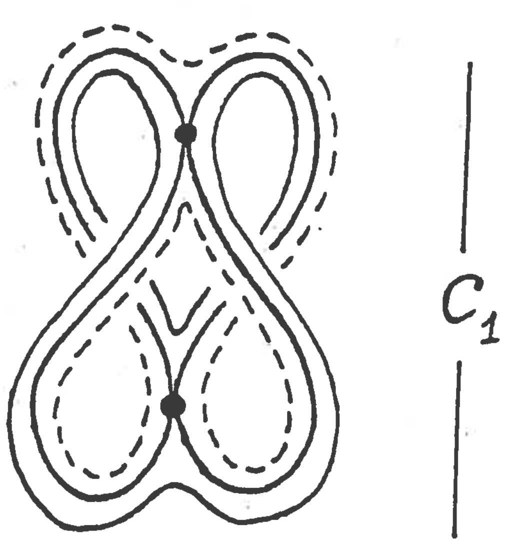


Fig. 28

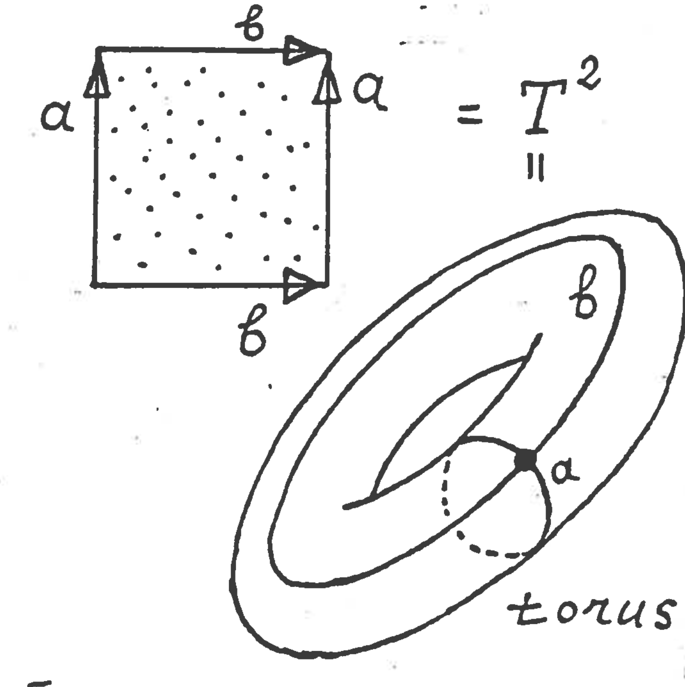


Fig. 32

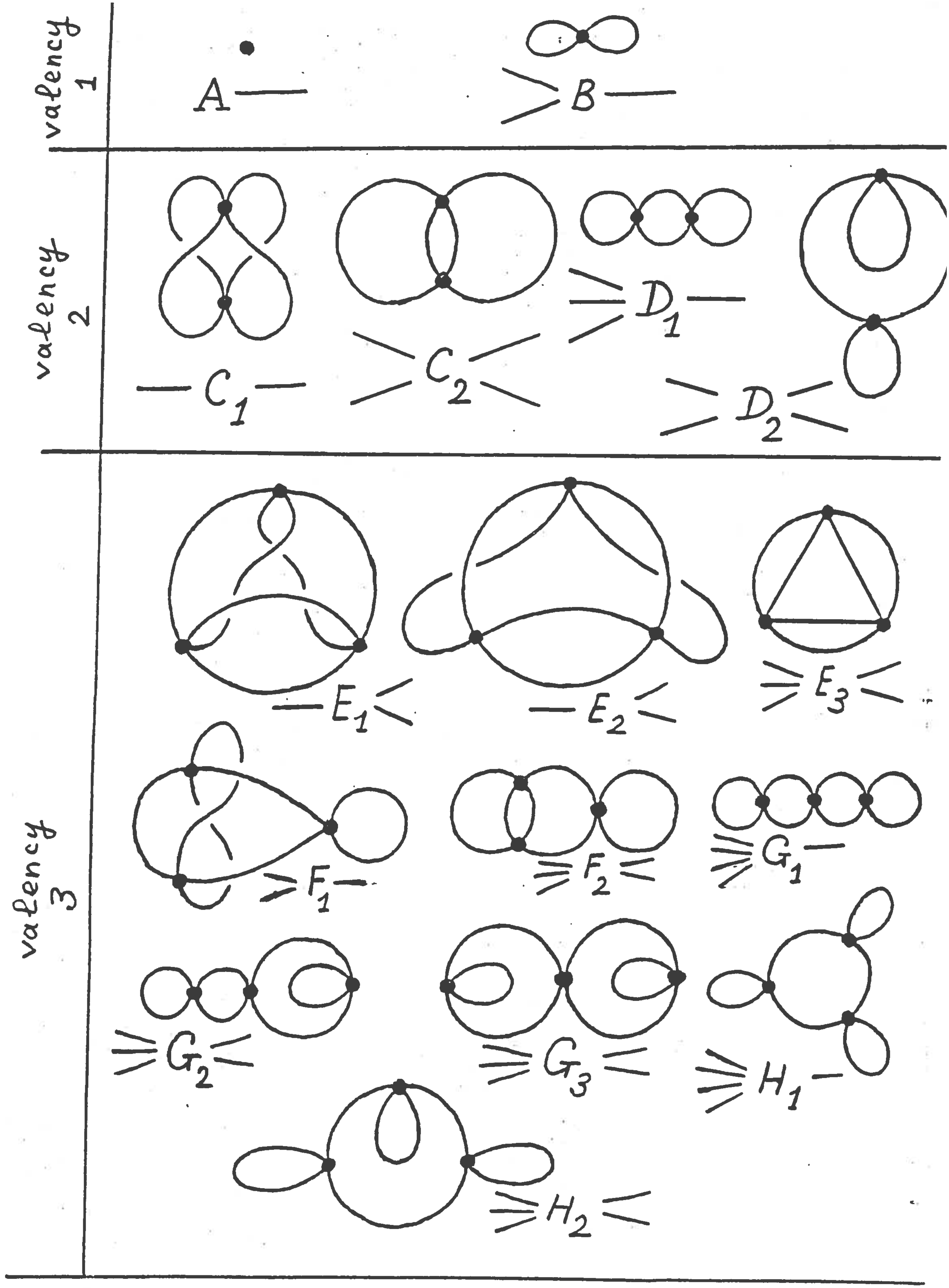

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Fig 30

Fig. 31-a

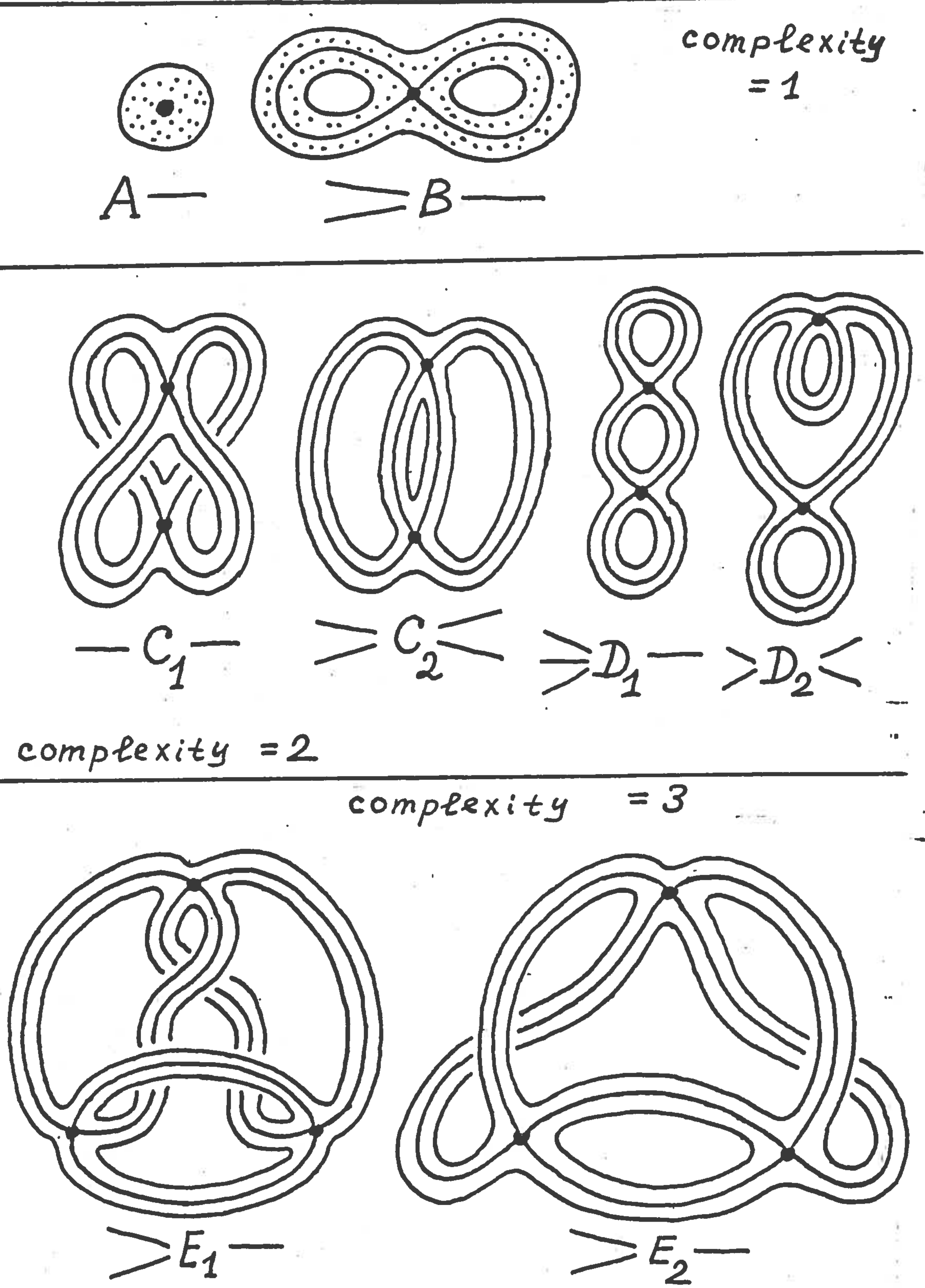
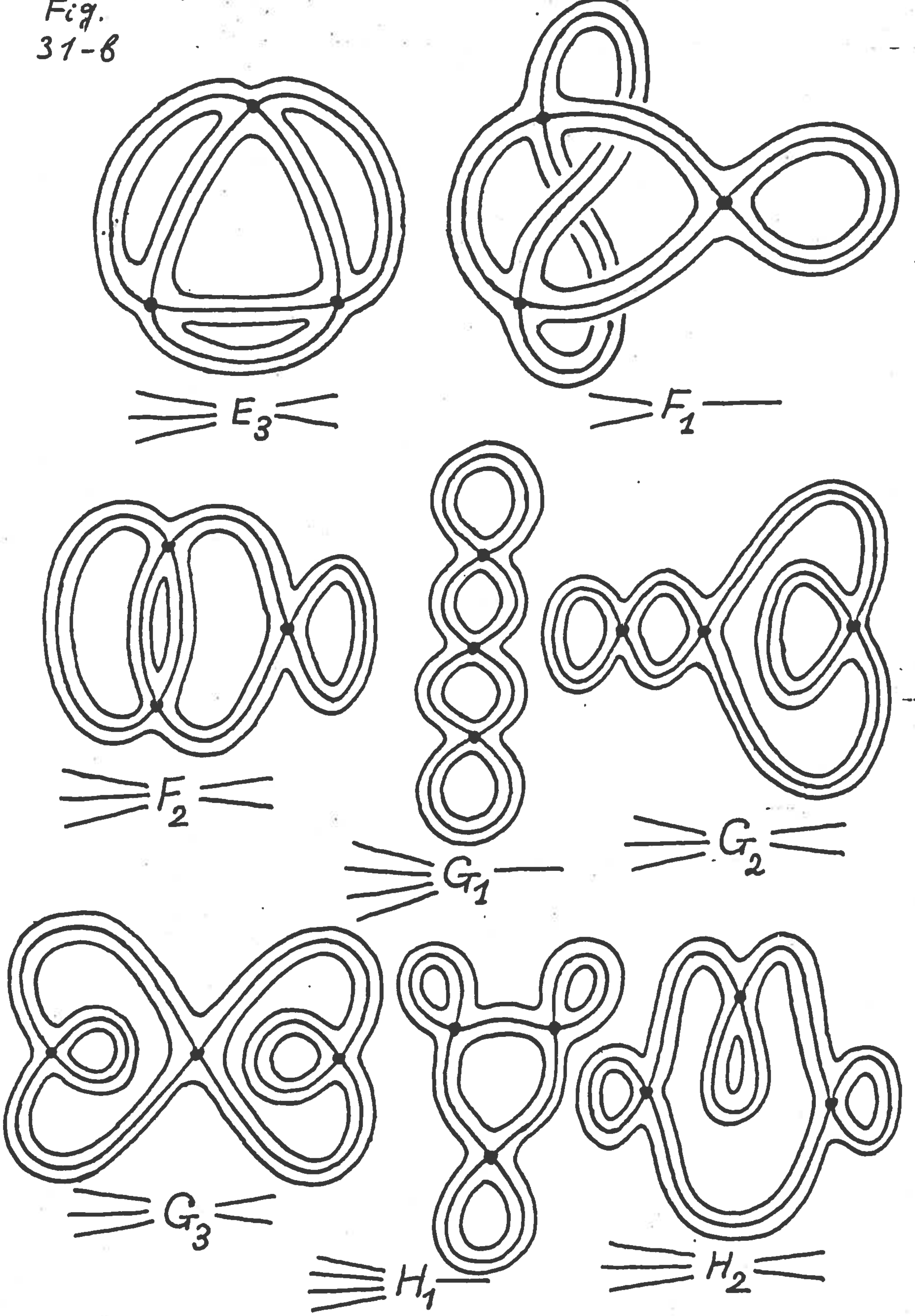


Fig.
31-в



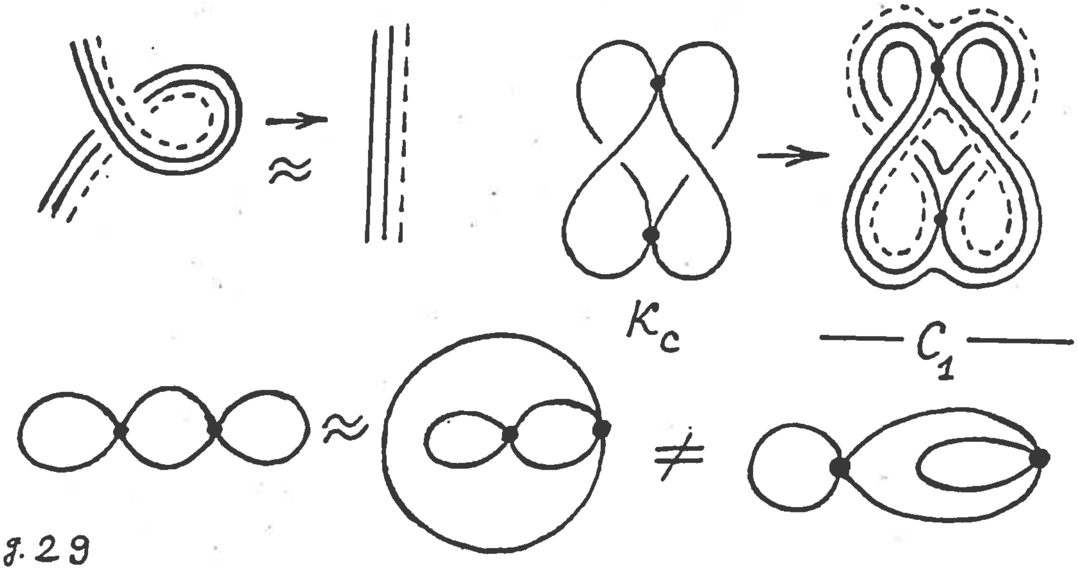
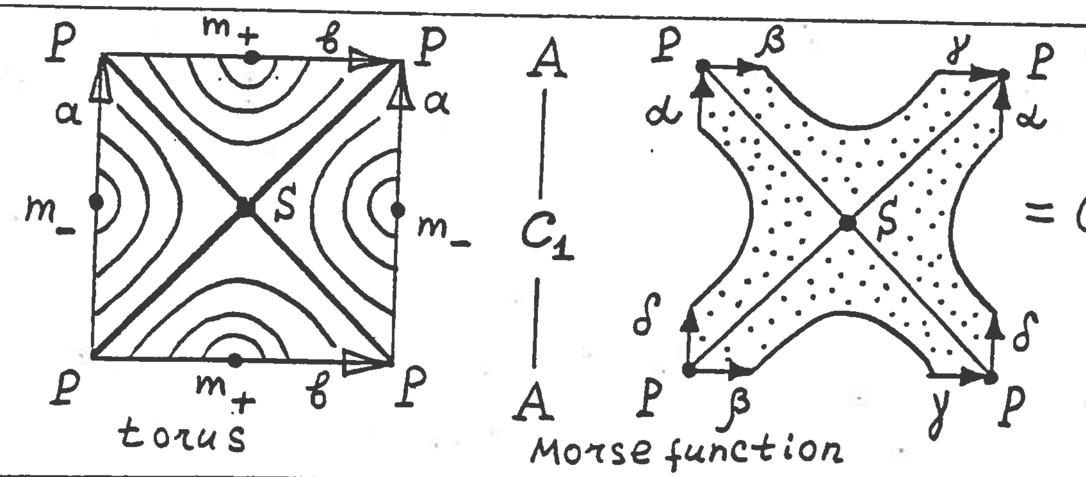


Fig. 29



Fi 33

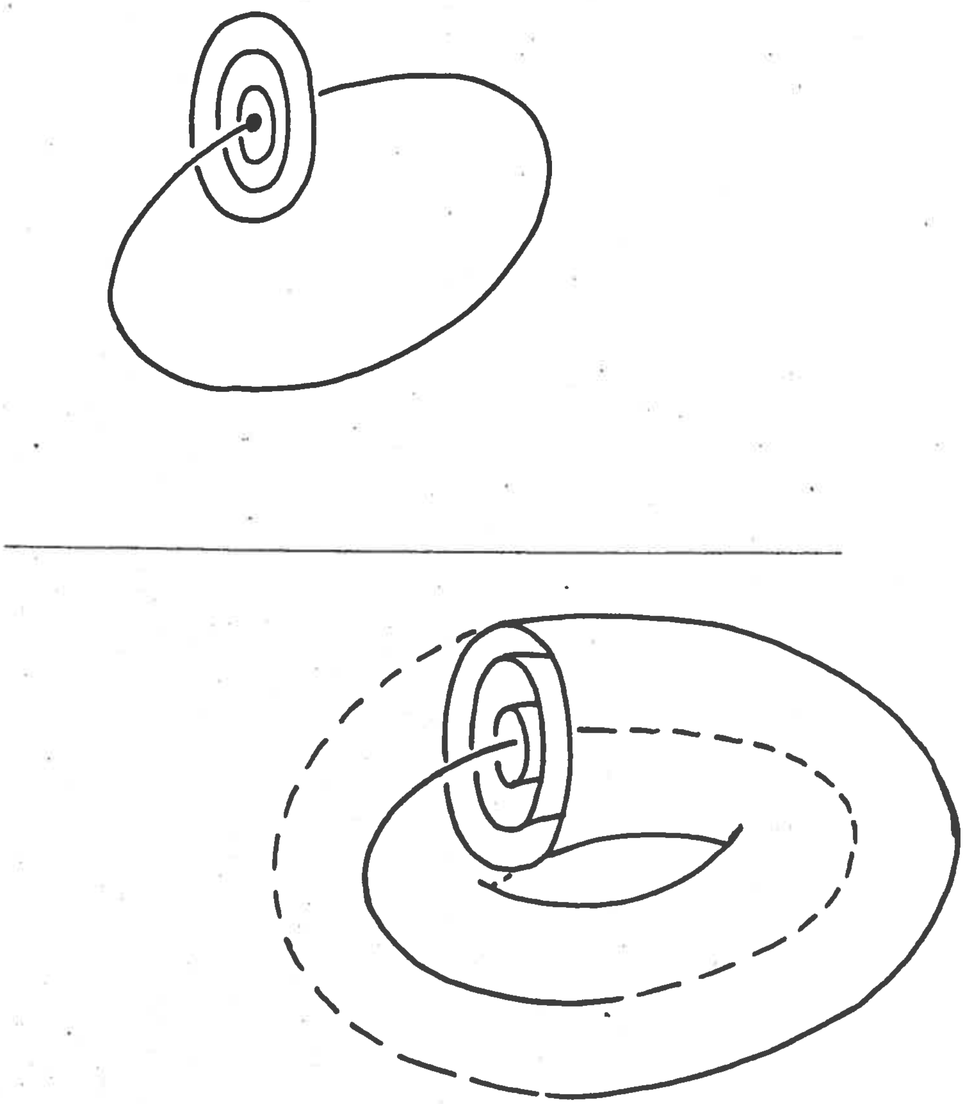


Fig. 34