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NONLINEAR STRUCTURES: SOLITON MODEL OF TURBULENCE

by

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Nonlinear Structures: Soliton Model of Turbulence

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Introduction. This paper is devoted to the problem of the strong Langmuir turbulence which combines both turbulent behavior and nonlinear structures. First such a model had been proposed in [1], futher results were presented in [2, 3].

In many problems of plasma physics, solid state physics, hydrodynamics of the surface waves etc. the weak turbulence approach may be used. It is based, as a rule, on the random phase approximation, thus reminding the incoherent light. The most typical window of parameters they use to determine the limits of its applicability, may be determined by the 'degree of turbulence' W/nT where W is the energy density of oscillations and nT the thermal energy density. For example, for the Langmuir waves, the most typical plasma mode, they use the following chain of inequalities:

$$1 \gg \frac{W}{nT} \gg \frac{1}{N_D}.$$

Here $N_D \gg 1$ is the number of particles in a sphere of the radius equal to the Debye length r_D. Only great N_D typical of the hot and/or rarefied plasmas allow the predomination of the colective effects.

Meanwhile, only in very few cases of the strong turbulence the left inequality becomes violated, so that $W \sim nT$. E.g., in hydrodynamics, it would mean the oscillating velocity being of the order of the acoustic velocity. Much more typical of strongly turbulent regimes are the effects of coherence, phase correlation and, after all, the nonlinear structures. Thus, the question arises: can such a regime be turbulent or not? The matter is that superposition principle breaks in essentially nonlinear media (cf., e.g., steady magnets), thus, at first sight, turbulent behavior becomes impossible.

This is the general problem, however, the physical community started studying strong Langmuir turbulence following, first of all, the problems of laser fusion. The typical mechanism of the energetic input into the plasma corona of a laser target was collective one even for Nd lasers ($\lambda = 1.06\,\mu$) and moreover for CO_2 lasers ($\lambda = 10.6\,\mu$). The collective absorption was being based on the parametric instabilities, in a result, the main fraction of energy put in turned out to transfer just to the Langmuir waves. The latters, within the frames of the weakly turbulent regime, would loose their energy, in average, little by little. Let us follow the dispersive relation of the Langmuir waves,

$$\omega = \omega_{pe}[1 + \frac{3}{2}(kr_D)^2], \quad kr_D \ll 1$$

with $\omega_{pe} \equiv (4\pi ne^2/m)^{1/2}$ being the electron plasma frequency and r_D Debye radius. One can readily see that the overwhelming fraction of the oscillatory energy remains frozen in the long-wave plasma waves with $\omega \simeq \omega_{pe}$. There is the only mechanism of damping of these waves, and not so efficient one, that is collisional damping. It is interesting to note that such a scenario turns out to be opposite to that of the conventional Kolmogorov turbulence, in which the source in the $k-$space corresponds to the longer scales while the leakage of the energy of oscillations is usually located in small space scales.

At first sight, that means the resulting accumulation of the wave energy in a plasma corona, that is so-called condensation of plasmons. In fact, nonlinear effects join the game, first of all, modifying the dispersion law:

$$\omega = \omega_{pe}[1 + \frac{3}{2}(kr_D)^2 - \beta\frac{W}{nT}], \tag{1}$$

where $\beta \sim 1$ and W is the energy density of the plasma waves. It is interesting to compare (1) with the relativistic form of the energy of a nonrelativistic particle:

$$\mathcal{E} = mc^2 + \frac{p^2}{2m} + U(\mathbf{r}). \tag{2}$$

Weak turbulence is nothing but the perturbation theory based on the zero level representation of the oscillating field as an ensemble of the non-interacting waves (quasi-free plasmons). Let us note that usual particles in (2) remain almost free and their energy may be estimated as $p^2/2m$ only if $U \ll p^2/2m$. It is no use to compare their potential energy with mc^2. Respectively, the validity of the weak turbulence theory for Langmuir plasmons should be estimated as

$$W \ll nT(kr_D)^2. \tag{3}$$

If this inequality becomes violated, intermode coupling turns out to be strong even on the level of zero order approximation. Although such a turbulence is not strong in the sense of separation of oscillating and random particle motion, the quasiparticles have to be built on the base of the renormalized theory. Thus, in fact, the parameter of expansion while constructing the weak turbulence theory has to be not W/nT but $W/[nT(kr_D)^2]$. First, it was established by Vedenov and Rudakov, 1965 (see, e.g., [2,3]). In particular, it was shown that, as a result of violation of (3) inequality, the specific modulational instability had to start resulting in the localization of plasmons in some clots or drops. In other words, instead of a homogeneous weak turbulence, nonlinear structures would arise.

Basic Equations. Langmuir Soliton. What does occur after weakly turbulent treatment becomes broken? It looks not incredible that we can proceed something like the quasiparticle formalism. However, the quasiparticles themselves have to be chosen of the new form and with some new properties. Plane wave (or another linear approach) does not fit more. Of course, Fourier expansion may be used in any case. But in general we have to study plane wave ensemble including fast varying phases, not only spectral intensities. To escape the violation of the main basic property of the stationarity of numbers of particles in each initial and final state we should search for a new kind of quasiparticles. Let us remind that modulational instability results in the localization of the oscillatory field, in other words, chaotic turbulence tends to the transformation into the nonlinear structures. Thus, it seems reasonable to start from the nonlinear equations in the x-space without using the Fourier transform. To separate oscillating and slow evolution of all the physical parameters, the following substitution is useful to use:

$$\vec{\mathcal{E}}(\mathbf{r},t) = \frac{1}{2}[\mathbf{E}(\mathbf{r},t)\exp(-i\omega_{pe}t) + c.c.] \tag{4}$$

with $\vec{\mathcal{E}}(\mathbf{r},t)$ being the complex amplitude and t the 'slow' time, i.e., $\partial/\partial t \ll \omega_{pe}$. The nonlinear dynamics turns out to obey Zakharov equations [4]:

$$\mathrm{div}[2i\frac{\partial \mathbf{E}}{\partial t} + 3\omega_{pe}r_D^2\nabla\mathrm{div}\mathbf{E} - \omega_{pe}\frac{\delta n}{n}\mathbf{E}] = 0 \tag{5}$$

$$(\frac{\partial^2}{\partial t^2} - c_s^2\nabla^2)\delta n = \nabla^2\frac{|E|^2}{16\pi M_i} \tag{6}$$

where δn is the density perturbation, $c_s = \sqrt{T_e/M_i}$ ion acoustic velocity, all other terms are conventional. We keep the operator 'div' in the LHS of (5) since two first terms are the potential vectors but the third. To keep the correct space symmetry we have to keep 'div' but only in the 1-d case. In the linear approximation, the system (5,6) becomes splitted resulting in the independed acoustic motion with immaterial HF pressure in the RHS of (6), and linear dispersion of Langmuir waves, in accordance with (1).

For simplicity, only 1-d case will be considered in this paper. It is useful to note that 3-d dynamics is essentially different (see [2, 4]).

Let us assume a very slow motion when both electrons and ions are permitted to be described by the Boltzmann distribution:

$$n_e = n_0\exp(\frac{e\Phi}{T} - \frac{P_{HF}}{n_0T}) = n_i = n_0\exp(-\frac{e\Phi}{T}), \qquad P_{HF} \simeq \frac{|E|^2}{16\pi} \ll nT,$$

where Φ is the potential of the charge separation. Thus, HF pressure repells the electrons, they, in turn, pull out the ions, as a result, the self-consistent density well becomes formed in which the oscillating field is 'locked'. It is just the result of the modulational instability. Then one can exclude Φ:

$$e\Phi = P_{HF}/2n_0 \Longrightarrow \delta n_i \simeq -n_0\frac{|E|^2}{16\pi n_0 T}.$$

Together with (5), it results immediately in the nonlinear Schrödinger equation (NSE),

$$i\frac{\partial E}{\partial t} + \frac{3}{2}\omega_{pe}r_D^2\frac{\partial^2 E}{\partial x^2} + \omega_{pe}\frac{|E|^2}{32\pi n M_i}E = 0. \quad (7)$$

This equation has been studied well enough. It is known that it has the infinite set of integrals of motion. In particular, it means that no turbulence is permitted to exist within the framework of (7) but only the entirely determinate nonlinear dynamics. We have not to forget, however, that (7) is not more than the quasi-steady or essentially subsonic model of the Langmuir dynamics. It is interesting to note that essentially subsonic limit of Eqs (5,6) is the same NSE. Meanwhile, in one point this system is opposite to (7) since it includes hydrodynamic description (6) of the background (i.e., ions) which is opposite limit with respect to the Boltzmann distribution. Respectively, the same effect of the field localization follows from (5,6) but conditioned by the different mechanism. To wit, the ion well in this case is the consequence not of the potential hill but of the potential well through which ions are flowing faster and $\delta n < 0$ follows from the continuity of the ion flux. Both cases are presented in Fig.1. Let us turn to the exact solution describing this effect of localization.

Fundamental object in the strong Langmuir turbulence is called Langmuir soliton (Rudakov, 1972, see, e.g., [2,3]). It can be obtained analytically starting from Eqs (5,6). We will search for this solution in a form of travelling wave:

$$E(x,t) = E(x - v_s t)exp[i(kx - \delta\omega t)], \qquad \delta\omega \equiv \omega - \omega_{pe}, \quad E_{x\longrightarrow\pm\infty} \longrightarrow 0. \quad (8)$$

Here v_s is the soliton velocity (not acoustic velocity c_s), the frequency shift $\delta\omega$ includes both dispersive and nonlinear effects. Space modulation (k) is inevitable if $v_s \neq 0$, as it will be seen below.

It is convenient, for simlicity, to start with substitution of (8) in NSE which turns out to be splitted in two:

$$\frac{3}{2}\omega_{pe}r_{De}^2 E_{\xi\xi} + (\delta\omega - \frac{3}{2}\omega_{pe}k^2 r_{De}^2)E + \omega_{pe}\frac{|E|^2}{32\pi n M_i}E = 0 \quad (9)$$

$$-v_s E_\xi + 3k\omega_{pe}r_{De}^2 E_\xi = 0 \quad (10)$$

where $\xi = x - v_s t$. (10) immediately results in

$$v_s = 3k\omega_{pe}r_{De}^2 \equiv \frac{\partial\omega^l(k)}{\partial k} \quad (11)$$

and that is just the argument for the space modulation. Indeed, the nonlinear wave velocity turns out to be equal to the group velocity of the Langmuir waves, hence, $v_s \neq 0 \Longrightarrow k \neq 0$. The difference within the brackets in the LHS of (9) is the nonlinear shift of the frequency while the total shift may be presented in the form

$$\delta\omega = \frac{3}{2}r_{De}^2(k^2 - k_0^2)$$

where k_0 will be found while solving (9), together with $E(\xi)$ dependence:

$$E = \frac{E_0}{\cosh k_0 \xi}, \qquad k_0 = \frac{eE_0}{\sqrt{24T}}. \tag{12}$$

This is the only nonlinear formation stable with respect to the modulational instability, and consequently, the final result of this instability. Unlike KDV solitons, it includes HF modulation and depends on two free parameters, viz., E_0 and v_s (or k). If, instead of NSE, one solves the full system of Zakharov equations (5,6), exact solutions will be slightly different from (12) due to the 'relativistic' effects:

$$E = \frac{E_0}{\cosh k_0 \xi}, \; \xi = x - v_s t, \; \delta\omega = \frac{3}{2} r_{De}^2 (k^2 - k_0^2), \; k_0 = \frac{eE_0}{\sqrt{24}(1 - v_s^2/c_s^2)T}. \tag{13}$$

It is interesting to consider the Fourier spectrum of the Langmuir soliton. Funnily enough, it turns out to be presented by the same function $\cosh^{-1}$:

$$E(x,t) = \mathrm{Re} \int_{-\infty}^{+\infty} dk' \frac{E_0}{2k_0 \cosh(\pi k'/2k_0)} \exp[i(k' + k)x - i(\omega + k' v_s)t] \equiv$$

$$\equiv \mathrm{Re} \int_{-\infty}^{+\infty} dq E_q \exp[i(qx - \Omega_q t)],$$

$$E_q = \sqrt{6}\sqrt{1 - v_s^2/c_s^2} \frac{T}{e\cosh[\pi(q - k)/2k_0]}, \qquad \Omega_q = \omega_{pe} + \delta\omega + (q - k)v_s. \tag{14}$$

Fourier spectra of both standing and travelling Langmuir solitons are drawn in Fig.2. At least two their interesting properties should be pointed out:

1) The amplitude of the spectral distribution in the k–space does not depend of E_0 but only the spectral width. This is the evident consequence of the fundamental relation $k_0 \propto E_0$.

2) $(\Omega_q - \Omega'_q)/(q - q') \equiv v_s$. This is a manifestation of the nature of the fundamental nonlinear processes involved into the problem:

$$l \longrightarrow l + s, \qquad l + l \longrightarrow (s) \longrightarrow l + l.$$

Thus, the low–frequency component of the Langmuir soliton (see δn in Fig.1) is the commom beat of all the HF harmonics.

Soliton model of the strong Langmuir turbulence. First of all, let us emphasize that, unlike NSE, not the infinite number of integrals of motion may be introduced for Zakharov equations (5,6) but only three, to wit,the number of quanta:

$$I_0 = \frac{1}{4\pi\omega_{pe}} \int_{-\infty}^{+\infty} |E|^2 dx, \tag{15}$$

the integral of momentum I_1, and the dispersive fraction of the oscillatory energy (like $\mathcal{E} - mc^2$ for the particles):

$$I_2 = \mathcal{E} - \omega_{pe} I_0. \tag{16}$$

As a result, Eqs (5,6) are not completely integrable and the turbulent behavior is allowed within the frames of the model based on (5,6).

Eqs (5,6) can describe both modulational instability and essentially nonlinear dynamics. As the modulational instability is conditioned by the level of turbulence high enough, $W/nT \geq (kr_D)^2$, one could expect its final result to be, at least, $W/nT \simeq (k_0 r_D)^2$ where k_0 is an effective wave number corresponding to some typical scale of localization, $L \sim k_0^{-1}$. If $W \longrightarrow nT$ due to the pumping (laser or particle beam or something else), effective damping process joins the game, that is Landau damping since $k_0 r_D \longrightarrow 1$ providing the dissipation. Thus, the flux of energy in the $k-$space becomes inverted with respect to the weak turbulence, and all the turbulent scenario acquires the typical features of the Kolmogorov–like turbulence.

As it has been noticed above, to represent any turbulent behavior, the superposition principle has to be provided by the model being used. From this point, Langmuir solitons are looking rather attractive to play the role of new quasiparticles since these coalescenes are restricted in space with exponential accuracy, hence, the superposition principle can be satisfied with the same accuracy. Besides, except of the amplitude, each soliton has one more free parameter, i.e., velocity, that allows to form real chaotic behavior of the resulting field. In addition, solitons of different amplitudes, have also different width, thus, one of them seems to be something like quasiclassical well for another, as a result, these solitons may pass free one through another, like KDV solitons do. After all, unlike any other wave formation, soliton is stable with respect to the modulational instability.

Following Eqs (12,13), to wit, $k_0(E_0)$ dependence, one readily can see that self consistent relation $W/nT \simeq (k_0 r_D)^2$ is true for any particular soliton. Taken as averaged in space, $< W > /nT$ may even essentially less than $(< k_0 > r_D)^2$, thus, to differ weak turbulence from the strong one, it is useful to follow the direction of the energy flux in the $k-$space.

The fundamental assumption was made in [1] that this flux in the strongly turbulent regime was provided by the soliton fusion 'two in one' in which only solitons with close amplitudes could take part. Indeed, integral of motion I_0 allows this process (and return process as well since $I_0 \propto E_0$). In turn, integral of motion I_2 allows the process of fusion but forbids the process of decay, 'one in two'. As for the close amplitudes, this assumption was made to escape the multi–soliton collaps, based on the idea of quasiclassical approach mentioned above. Both assumptions were confirmed later in the 'computer experiment' carried out by Degtyarev et al [5]. Its results added only one but essential circumstance to the model concerning the important role of the acoustic waves (background noises) in the dynamics of multi–soliton systems. The qualitative picture of the coalescence 'two in one' following from these simulations is presented in Fig.3.

Let us put W to be the average density of the turbulent state and L — the length of the 1-d turbulent plasma system. Then let us introduce the set of the fundamental states in assumption that in each state the turbulent energy is distributed between N identical solitons so that N is the parameter of state. Their amplitude may be determined by using the equality

$$WL = N\mathcal{E}_0(N) = \frac{\sqrt{6}}{2\pi e}E_0(N)NT. \tag{17}$$

The maximal number of solitons in a state is restricted by the condition of close packing $N_{max} \simeq k_0 L$, which yields

$$N_{max} \simeq \frac{1}{2\sqrt{6}}(\frac{W}{nT})^{1/2}\frac{L}{r_D}. \tag{18}$$

The minimal number of solitons is conditioned by the Debye scale, or, if there exists suprathermal 'tail' of electrons, by some $k_{0\,max} \leq r_D^{-1}$ which determines the cut off the turbulent spectrum:

$$N_{min} \simeq \frac{1}{24}\frac{W}{nT}\frac{L}{r_D}(k_{0\,max}r_D)^{-1}. \tag{19}$$

In a result, the turbulent state may be presented in the form of expansion over fundamental states. Let us define $P(N)$ as the probability of (N) state:

$$P(N) = \frac{\Delta n(N)}{N} \tag{20}$$

where $\Delta n(N)$ is the number of solitons of the amplitude $E_0(N)$ in the real state. Hence, $P(N)$ is also the fraction of the total energy provided by these solitons as $\mathcal{E} \propto E_0$:

$$\Delta\mathcal{E}(N) = \mathcal{E}_0(N)P(N). \tag{21}$$

As a rule, in the computer simulations they follow the energetic spectrum in the $k-$space (the same had been studied in the Kolmogorov model). Well, let us calculate the spectral intensity W_k depending on the $P(N)$ distribution. For this purpose the expansion (14) will be used. In the turbulent state, all the solitons have to be placed randomly, with random phases. Thus, the squares of their harmonic amplitudes are allowed to be summarized:

$$W_k = \frac{12T^2}{Le^2}\int_{N(min)}^{N(max)} dN\, NP(N)\cosh^{-2}(\frac{3T^2Nk}{e^2WL}). \tag{22}$$

Roughly, for simplicity, $\cosh^{-2}(x)$ may be estimated by the step function $\Theta(1-|x|)$ which cuts off the integration at

$$N_0 \simeq \frac{1}{3}(\frac{e}{T})^2\frac{WL}{k} < N_{max}.$$

Thus, the final result may be presented as

$$W_k = \frac{12T^2}{Le^2} \int_{N(min)}^{N(0)} dN\, N P(N). \tag{23}$$

In many cases, spectra of strong Langmuir turbulence obtained in simulations may be well approximated by the function

$$W_k \propto k^{-2}$$

which in our representation corresponds to

$$P(N) = \text{const} \tag{24}$$

thus being an analog of the Rayleigh–Jeans distribution. (In a weakly turbulent regime, i.e., in the Fourier representation, this distribution degenerates to $W_k = \text{const} \equiv 2T/\pi$).

Our representation may be without difficulties translated into more usual treatment, operating not with the probability $P(N)$ but with the averaged number of solitons of the given amplitude per unit length, i.e., with the soliton distribution function:

$$F(E_0) = P(N)N\frac{dN}{dE_0}. \tag{25}$$

Particularly, 'Rayleigh–Jeans distribution' given by (24) becomes transformed into the following:

$$F(E_0) \propto E_0^{-3}.$$

Conclusion. Thus, it has been shown that strong turbulence of the plasma waves combines two basic properties of the nonlinear dynamics, viz., turbulent behavior and nonlinear structures. The latters can be modelled in one dimension by specific two–parametric solitons with HF modulation. Perhaps, this model can be expanded, in principle, onto some other nonlinear dynamics based on the interaction of individual objects with some individual properties resulting in the chaotic behavior that, in turn, results in some macroscopic dynamics. This dynamics includes the irreversible processes and may be contemplated, in a whole, as the complicated dynamical dissipative structure.

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Figure captions

Fig.1. Langmuir soliton

Fig.2. Fourier spectra of both immovable (solid line) and travelling (dotted line) Langmuir solitons

Fig.3. Three stages of the process of fusion of two Langmuir solitons

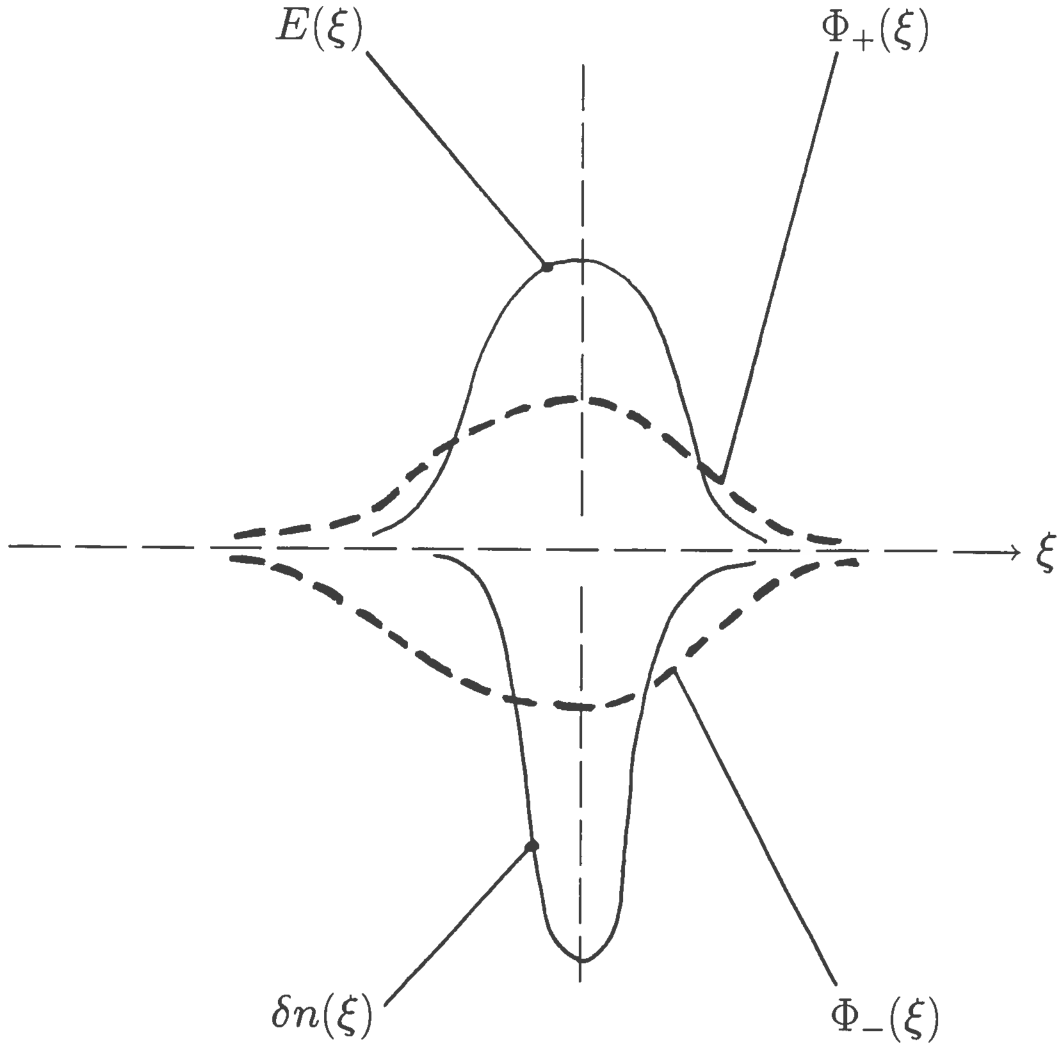

Fig.1

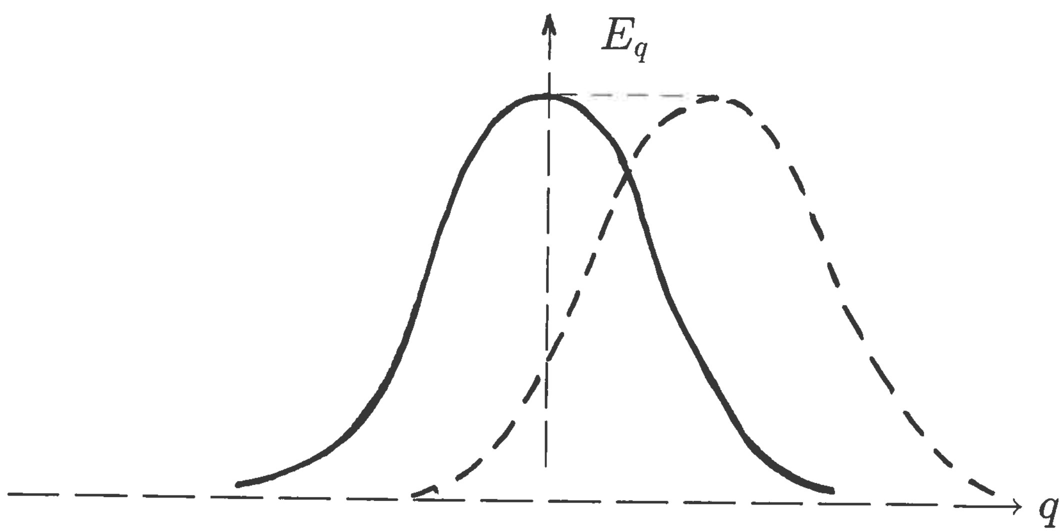

Fig.2

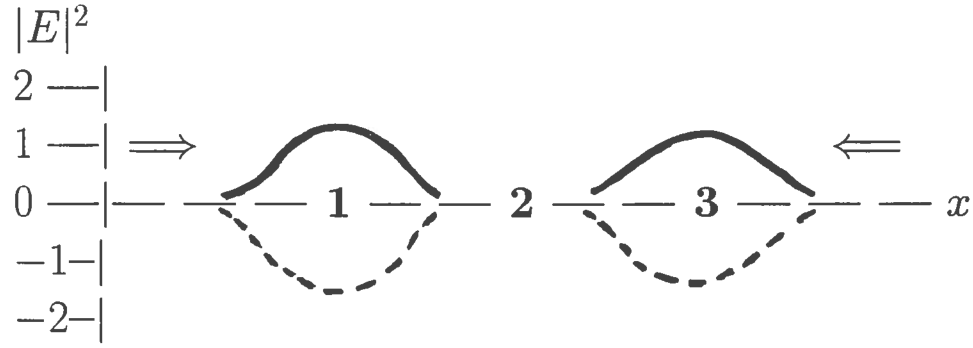

Fig.3a

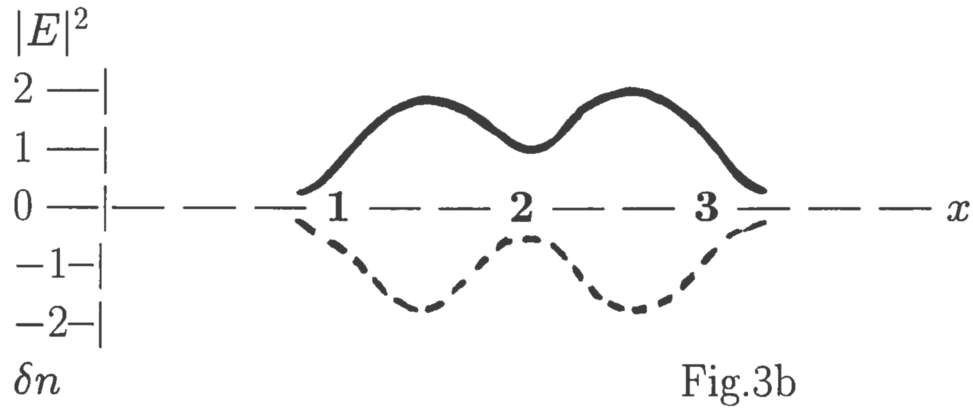

Fig.3b

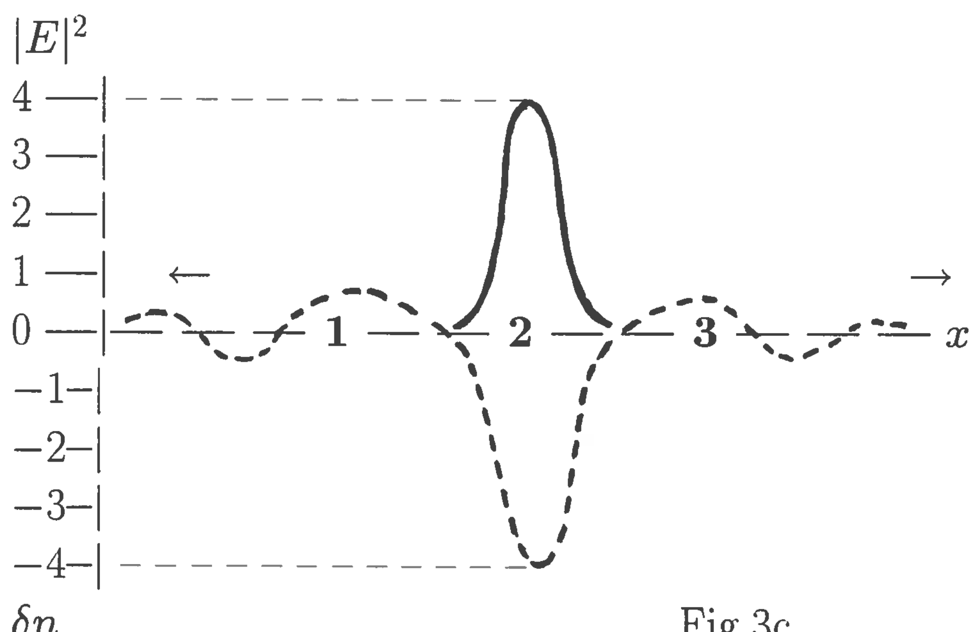

Fig.3c